Mathematical Reviews

Edited by

O. Neugebauer M. H. Stone O. Veblen

R. P. Boas, Jr., *Executive Editor*

Vol. 7, No. 4 April, 1946 pp. 145–184

TABLE OF CONTENTS

THE MICROFILM SERVICE

Through the services of the Brown University Photographic Laboratory one can obtain, either on microfilm or as photoprint, the complete text of any article listed in MATHEMATICAL REVIEWS or available in the Brown University Library, except books and material of which the reproduction is prohibited by copyright laws.

Procedure for Ordering Microfilm or Photoprint

1. Give your name and address.
2. State whether you wish microfilm or photoprint.
3. (a) If the article in question has been listed in MATHEMATICAL REVIEWS, give the MF number printed at the end of the title. Titles without MF numbers are copyrighted.

   (b) If the article has not been reviewed (for instance because published prior to 1939) give as accurately as possible the author, title, journal, volume, year and pages.
4. Enclose cash for the amount of the order. U. S. currency or check, made payable to Brown University, will be accepted. The minimum charge is 45¢.
5. Mail to: *Brown University Photographic Laboratory, Providence, Rhode Island.*

Cost

MICROFILMS are furnished for

15¢ for articles with less than 25 pages.

30¢ for articles with more than 24 pages, but less than 50 pages.

50¢ for articles with more than 49 pages, but less than 100 pages.

$1.00 for articles with more than 99 pages.

PHOTOPRINTS cost 20¢ per sheet, each sheet containing two pages of text, except in the case of unusually large journals (larger than 7" by 10"). In the latter case you will be informed of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price of 20¢ per page.

MINIMUM CHARGE. If an order totals less than 45¢ the cash included should be raised to 45¢ in order to cover the basic expenses of this service.

Microfilm and photoprints with MF numbers show black writing on white background. Articles which are not reviewed, and which therefore do not have an MF number, are reproduced only in negative films or prints, i.e. white letters on black background. Microfilms can be read in any reading machine adapted to standard 35 mm. double perforated film. Photoprints are on sheets 10" by 14" (about 25 by 35 cm.).

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

An edition of MATHEMATICAL REVIEWS printed on only one side of the paper is available to persons interested in making card files of reviews or wishing to add remarks to reviews in the future. This special edition may be obtained for an additional payment of $1.00. A regular current subscription can be changed to a one-side subscription by paying the additional $1.00. This edition is folded but not stitched.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
THE EDINBURGH MATHEMATICAL SOCIETY
ACADEMIA NACIONAL DE CIENCIAS EXACTAS, FISICAS Y NATURALES DE LIMA
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
UNION MATEMATICA ARGENTINA

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Ve

Ch

the

Ch

(N
pr
of
fo
wh
ha
14

M

27

Ea

al
au
hi
ce
nu

Ka

N
wh
Th
wi
pr

th
ac
re

V

b,
fo

Mathematical Reviews

Vol. 7, No. 4 APRIL, 1946 Pages 145-184

NUMBER THEORY

Chowla, S., and Singh, Daljit. A perfect difference set of order 18. Proc. Lahore Philos. Soc. 7, 52 (1945). [MF 14278]

The set was described in Math. Student 12, 85 (1945); these Rev. 6, 259.

Chowla, S. On $g(k)$ in Waring's problem. Proc. Lahore Philos. Soc. 6, 16–17 (1944). [MF 14329]

Continuing work of Rubugunday [J. Indian Math. Soc. (N.S.) 6, 192–198 (1942); these Rev. 5, 142] the author proves that 2^k-1 is not a factor of 3^k+1 if k is a multiple of 7, or a multiple of 5 but not of 75. He states that "the formulae of Pillai and Dickson determine $g(k)$ exactly except when 2^k-1 is a factor of 3^k+1." However, the reviewer has evaluated $g(k)$ in this case [Amer. J. Math. 66, 137–143 (1944); these Rev. 5, 142]. *I. Niven.*

Mian, A. M., and Chowla, S. The differential equations satisfied by certain functions. Proc. Lahore Philos. Soc. 6, 9–10 (1944). [MF 14325]

Identical with a paper in J. Indian Math. Soc. (N.S.) 8, 27–28 (1944); these Rev. 6, 119.

Erdös, P. On highly composite numbers. J. London Math. Soc. 19, 130–133 (1944). [MF 13631]

The number n is called highly composite if $d(m)<d(n)$ for all $m<n$, where $d(n)$ is the number of divisors of n. The author proves that, if n is highly composite, then the next highly composite number is less than $n+n(\log n)^{-c}$ for a certain c. This shows that the number of highly composite numbers not exceeding x is greater than $(\log x)^{1+c}$.
B. W. Jones (Ithaca, N. Y.).

Kantz, Georg. Über einen Satz aus der Theorie der biquadratischen Reste. Deutsche Math. 5, 269–272 (1940). [MF 14331]

Let $n=i^\rho a^\alpha b^\beta c^\gamma\cdots$ be a Gaussian integer with odd norm N, and m any complex integer prime to n. Let $R_1, R_2, \cdots, R_\nu$, where $\nu=\frac{1}{4}(N-1)$, be a quarter-residue system (mod n). Then the system $mR_1, \cdots, mR_\nu$ coincides (except for order) with the system $\epsilon_1R_1, \cdots, \epsilon_\nu R_\nu$ (mod n), where the ϵ_k are properly chosen powers of i; and finally

$$\epsilon_1\epsilon_2\cdots\epsilon_\nu=\left[\frac{m}{n}\right],$$

the generalized Eisenstein symbol for the biquadratic character of m with respect to n. For n a complex prime this reduces to a theorem of Gauss. *L. Carlitz.*

Vandiver, H. S. Bernoulli's numbers and certain arithmetic quotient functions. Proc. Nat. Acad. Sci. U. S. A. 31, 310–314 (1945). [MF 13438]

The author has defined the generalized Bernoulli number $b_n(g, h)=(bg+h)^n$, where in this expansion b_k is substituted for b^k, and where b_k is the ordinary Bernoulli number defined symbolically by $(b+1)^n=b^n$ $(b^k=b_k)$ [Duke Math. J. 8, 575–584 (1941); these Rev. 3, 67]. In this paper he derives the following result. If p is an odd prime with $p-1=mc$, c even and $g\not\equiv 0 \pmod p$, then

$$p^{-1}\sum_r (gr+h)^n\equiv -cn\sum_{k=0}^{m-1}\frac{b_{kc+n}(g, h)}{kc+n} \pmod p$$

for n even and not divisible by c, where the summation on the left extends over all integers r in the set $0, 1, \cdots, p-1$ such that $(gr+h)^c\equiv 1 \pmod p$. For the special case $g=1$, $h=0$, this becomes a result about ordinary Bernoulli numbers which was proved by the author [same Proc. 31, 55–60 (1945); these Rev. 6, 170]. *H. W. Brinkmann.*

Gloden, A. Sur la congruence $X^4+1\equiv 0 \pmod p$. Bull. Soc. Roy. Sci. Liége 12, 390–392 (1943). [MF 13149]

Gloden, A. Table des solutions de la congruence $X^4+1\equiv 0 \pmod p$ pour $2.10^5<p<3.10^5$. Mathematica, Timişoara 21, 45–65 (1945). [MF 13972]

The first paper is an announcement of the completion of the table contained in the second paper, together with a brief explanation (reproduced in the second paper) of the method of construction. The primes p for which the congruence

$$X^4+1\equiv 0 \pmod p \tag{1}$$

is possible are those of the form $8k+1$. They are expressible in essentially one way by the forms $p=x^2+y^2=2z^2+t^2$. If one solves for u and v the linear congruences

$$x\equiv uy \pmod p, \quad z\equiv vt \pmod p$$

then the four solutions of (1) are $\pm v(u\pm 1)$. The table (pp. 47–65) gives two solutions $v(u\pm 1) \pmod p$ for each of the 1968 primes of the form $8k+1$ between 200000 and 300000. As a by-product the complete new factorizations of 24 numbers of the form a^4+1 are given. This is an extension of two previous tables by A. J. C. Cunningham [Binomial Factorisations, vols. 1, 4, London, 1923] for $p<100000$, and S. Hoppenot [Tables des Solutions de la Congruence $x^4\equiv -1 \pmod N$ pour $100000<N<200000$, Brussels, 1935]. *D. H. Lehmer.*

Pillai, S. S. On the equation $2^x-3^y=2^X+3^Y$. Bull. Calcutta Math. Soc. 37, 15–20 (1945). [MF 13303]

The following Diophantine equations are solved completely:

$$2^x-3^y=3^Y-2^X, \quad 2^x-3^y=2^X+3^Y, \quad 3^y-2^x=2^X+3^Y.$$

A. Brauer (Chapel Hill, N. C.).

Størmer, Carl. Sur un problème curieux de la théorie des nombres concernant les fonctions elliptiques. Arch. Math. Naturvid. 47, no. 5, 83–85 (1943). [MF 12975]

The author studies the Diophantine equation

$$m\int_x^\infty \frac{dt}{(4t^3-g_2t-g_3)^{\frac{1}{2}}}+n\int_y^\infty \frac{dt}{(4t^3-g_2t-g_3)^{\frac{1}{2}}}=k\omega,$$

where 2ω is the real period of the corresponding Weierstrass $\wp$-function. He obtains 9 solutions in integers x, y, m, n, and k, but he is not able to prove that there exist no other solutions. *A. Brauer* (Chapel Hill, N. C.).

Teghem, Jean. Sur l'application de la théorie des sommes de Weyl à des problèmes d'inégalités diophantiennes. Bull. Soc. Roy. Sci. Liége 11, 4–6 (1942). [MF 13091]

The author gives, without proofs, a number of results improving considerably on earlier ones by J. F. Kokama [Over Stelsels Diophantische Ongelijkheden, Groningen thesis, 1930]. *K. Mahler* (Manchester).

Ollerenshaw, Kathleen. The critical lattices of a square frame. J. London Math. Soc. 19, 178–184 (1944). [MF 13640]

Let F_μ be the region between the squares $(\pm 1, \pm 1)$, $(\pm\mu, \pm\mu)$, where $0<\mu<1$. A lattice is defined to be F_μ-admissible if none of its points is in the interior of F_μ. If $\Delta(F_\mu)$ is the lower bound of the determinants of F_μ-admissible lattices, a lattice whose determinant is actually equal to $\Delta(F_\mu)$ is known as a critical lattice. The author proves that $\Delta(F_\mu)=[1/(1-\mu)]^{-2}$ and constructs all critical lattices. The result is applied to derive a lattice-point theorem for an arbitrary lattice with determinant Δ by magnifying F_μ so that its magnification has a critical lattice with determinant Δ. *D. Derry* (Saskatoon, Sask.).

Bell, E. T. A representation of certain integer powers. Nat. Math. Mag. 20, 3–4 (1945). [MF 13977]

The author proves the following theorem and other results related to it. If a and b are coprime positive integers and n is a positive integer, a positive integer s and positive integers ξ, η, ζ may be found such that $x=\xi$, $y=\eta$, $z=\zeta$ is a solution of $n^{as}=F_{3a+2b,2a+b}(x,y,z)/F_{b,3a+2b}(x,y,z)$, where $F_{r,t}(x,y,z)$ denotes the ternary form $x^ry^t+y^rz^t+z^rx^t$. *B. W. Jones* (Ithaca, N. Y.).

Griffiths, L. W. Universal functions of polygonal numbers. III. Amer. J. Math. 67, 443–449 (1945). [MF 12925]

The author continues the investigation of those quadratic forms

$$f=\sum_{i=1}^{n} a_i(x_i+mx_i(x_i-1)/2), \qquad m\geqq 3,$$

which represent all integers. The paper is devoted to the case in which the weight $w=\sum_{i=1}^n a_i$ has the value $m+4$. The case $w\leqq m+3$ was treated in part II [same J. 66, 97–100 (1944); these Rev. 5, 199]. The results are too complicated to be quoted here. *D. H. Lehmer.*

Pry, G., et Prigogine, I. Rayons et nombres de coordination de quelques réseaux simples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 866–873 (1942). [MF 13684]

Suppose that a lattice contains N_k points at distance R_k from the origin, for $k=1, 2, \cdots$. In the case of the simple cubic lattice we have $N_k=\nu_k$, the number of ways of expressing k as a sum of three integral squares; for example, $\nu_{14}=48$ from the permutations of $(\pm 1)^2+(\pm 2)^2+(\pm 3)^2$. Here $R_k^2=k$ (and $R_1^2=1$). In the case of the body-centered cubic lattice, the N's are the nonvanishing terms of the sequence $\mu_3, \mu_4, \cdots$, where μ_q is the number of ways of expressing q as a sum of three squares all of the same parity as q itself; thus $\mu_{4n}=\nu_n$, $\mu_{8n+3}=\nu_{8n+3}$, but every other $\mu_q=0$. Here $R_k^2=q$ and $R_1^2=3$, so $R_k^2/R_1^2=q/3$. Finally, in the case of the face-centered (or close packed) cubic lattice we have $N_k=\nu_{2k}$, $R_k^2=2k$, $R_1^2=2$, $R_k^2/R_1^2=k$. The authors tabulate these sequences up to $k=25$, without mentioning their interdependence. They give the correct value $N_7=\nu_{14}=48$ for the close packed lattice, but the incorrect value 24 for ν_{14} and μ_{56} (cited as $R_k^2/R_1^2=18\frac{2}{3}$) in connection with the other two lattices. *H. S. M. Coxeter.*

Roussel, André. Sur le nombre des nombres premiers inférieurs à une valeur donnée. C. R. Acad. Sci. Paris 220, 842–844 (1945). [MF 14069]

Using a result of É. Picard expressing the number of solutions of the system $f(x,y)=0$, $g(x,y)=0$ in terms of a double integral, the author, by taking $f=\sin \pi x$ and $g=y-\sin\{\pi\Gamma(x)/x\}$, obtains an exact expression, involving a double integral over a rectangular area, for the number $\pi(n)$ of primes not exceeding n. *I. Niven.*

Wormser, Guy. Sur les nombres premiers représentables par des polynomes du second degré. C. R. Acad. Sci. Paris 220, 159–160 (1945). [MF 13490]

Using the terminology of a previous paper [same C. R. 217, 241–242 (1943); these Rev. 6, 37] and the notation $f_{n,p}(x,1)=f_{n,p}(x)$, the author proves the theorem: if $f_{n,p}(x)$ is a prime number for every integer x for which $0\leqq x\leqq z_{n,p}f_{n,p}(x)$ then (1) $f_{n,p}(x)$ is a prime number for all integers x such that $Z_{n,p}<x<p-1$; (2) if, for any integer q, $f_{n,p}(q)$ is prime to each of the numbers $f_{n,p}(i)$, $0\leqq i\leqq q-1$, then $f_{n,p}(q)$ is a prime number. *B. W. Jones.*

Linnik, U. V. On a theorem in the theory of primes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 7–9 (1945). [MF 13749]

The formula

$$\sum_{x=1}^{N^{\frac{1}{2}}}[\psi\{(x+1)^2\}-2\psi\{(x+\tfrac{1}{2})^2\}+\psi(x^2)]=O(N^{63/64}),$$

where $\psi(x)$ is Chebyshev's function from the theory of prime numbers, is proved by means of results due to Hoheisel [S.-B. Preuss. Akad. Wiss. 1930, 72–82 (1930)]; cf. Ingham [Quart. J. Math., Oxford Ser. 8, 255–266 (1937)] and Segal [Trav. Inst. Math. Stekloff 4, 37–48, 49–62 (1933)]. *H. D. Kloosterman* (Leiden).

Linnik, U. V. On the characters of primes. I. Rec. Math. [Mat. Sbornik] N.S. 16(58), 101–120 (1945). (English. Russian summary) [MF 13004]

This paper contains estimates for the sum $\sum_{p\leqq N}\chi(p)$, where $\chi(n)$ is either a complex primitive character (mod D) or $\chi(n)=(-D/n)$, where $-D<0$ is a fundamental discriminant. The fundamental lemma of the paper asserts that, if $\rho_k=\beta_k+it_k$ $(k=1, 2, \cdots)$ are the zeros of $L(s,\chi)$ in the critical strip, then there exists an absolute constant C such that

$$\sum m_k|\rho_k|^{-2}\exp(-C(1-\beta_k)\log D)$$

is bounded, where m_k is the multiplicity of the zero ρ_k. This paper precedes logically the author's papers on the least prime in an arithmetic progression [same Rec. N.S. 15(57), 139–178, 347–368 (1944); these Rev. 6, 260], where a knowledge of it is assumed. *H. Davenport* (London).

Siegel, Carl Ludwig. On the zeros of the Dirichlet L-functions. Ann. of Math. (2) 46, 409–422 (1945). [MF 13428]

The author considers the number of zeros of Dirichlet's function $L(s,\chi)$, where χ is a character with modulus m, in

rectangles $\frac{1}{2}+\delta<\sigma<1$, $-T_0<t<T_0$, or $0<\sigma<1$, $0\leqq t\leqq T_0$ (T_0 fixed). The number of zeros in the former is

$$O\{\phi(m)\delta^{-1}\log^{-20} m\}$$

if $\delta>1/\log\log m$, and the number in the latter is

$$(T/2\pi)\phi(m)\log m+O\{\phi(m)\log^{\frac{1}{2}} m\};$$

here $\phi(m)$ denotes the number of characters χ. Various consequences are noted, for example, that every point of the line $\sigma=\frac{1}{2}$ is a limit point for the set of the zeros of $L(s,\chi)$, with variable m and χ. On the Riemann hypothesis this was proved by the reviewer [Proc. London Math. Soc. (2) 32, 488–500 (1931)]. *E. C. Titchmarsh.*

Wintner, Aurel. Mean-values of arithmetical representations. Amer. J. Math. 67, 481–485 (1945). [MF 13936]

The author proves that every multiplicative function which is bounded and nonnegative has a mean value. That is, $\lim_{n\to\infty}\{f(1)+f(2)+\cdots+f(n)\}/n$ exists. *P. Erdös* (Stanford University, Calif.).

Chabauty, Claude. Démonstration nouvelle d'un théorème de Thue et Mahler sur les formes binaires. Bull. Sci. Math. (2) 65, 112–130 (1941). [MF 13265]

The author gives a new proof, independent of Diophantine approximation, for the following theorem of K. Mahler [Math. Ann. 107, 691–730 (1933)]. Let λ, μ, ν be three different algebraic integers generating a field K of degree k over the rational field R. There exists at most a finite number of pairs of rational relatively prime integers a, b such that the principal ideals $(a-\lambda b)$, $(a-\mu b)$, $(a-\nu b)$ have only a finite number of given prime factors $\mathfrak{p}_1,\cdots,\mathfrak{p}_h$ in K. The proof uses a $\mathfrak{p}$-adic method due to T. Skolem [Math. Ann. 111, 399–424 (1935)]. It runs as follows. Let the assertion be false. Then there exists an infinite sequence E of pairs of rational integers a, b with $(a,b)=1$ such that $(a-\lambda b)$, $(a-\mu b)$, $(a-\nu b)$ have only the prime factors $\mathfrak{p}_1,\cdots,\mathfrak{p}_h$. Without loss of generality, an infinite subsequence E_1 of E can be chosen in which $|a-\lambda b|_{\mathfrak{p}_i}\geqq c_0>0$, $|a-\mu b|_{\mathfrak{p}_i}\geqq c_0>0$, $|a-\nu b|_{\mathfrak{p}_i}\to 0$, $i=1,\cdots,h'$; $h'\leqq h$, and such that a/b tends to a limit, α say. Choose a prime number n greater than $4k(k^2+h+3)^2$ and greater than the exponents of the roots of unity contained in K. Then there is an infinite subsequence E_2 of E_1 such that $a-\lambda b=\varphi_0\varphi^n$, $a-\mu b=\psi_0\psi^n$ with constant integers φ_0, ψ_0 and variable integers φ, ψ in K satisfying $|\varphi|_{\mathfrak{p}_i}\geqq c_1>0$, $|\psi|_{\mathfrak{p}_i}\geqq c_1>0$, $i=1,\cdots,h'$. Denote by θ a root of $z^n=\psi_0(\lambda-\nu)/\varphi_0(\mu-\nu)$ not in K, and by H the field $H=K(\theta)$. Then an infinite subsequence E_3 of E_2 exists such that the principal ideal (Δ) in H generated by $\Delta=\varphi-\theta\psi$ is of the form $(\Delta)=\mathfrak{A}\mathfrak{P}_1^{l_1}\cdots\mathfrak{P}_j^{l_j}$, $l_1,\cdots,l_j$ nonnegative integers, where $\mathfrak{A}$ is one of a finite number of ideals and $\mathfrak{P}_1,\cdots,\mathfrak{P}_j$ are $j\leqq h$ fixed prime ideals in H. Put

$$\Omega(x,y;u,v,w;\zeta^g)=\sqrt[n]{(x-uy)(v-w)}-\zeta^g\sqrt[n]{(x-vy)(u-w)},$$

where ζ is a primitive nth root of unity, $g=0,1,\cdots,n-1$, and the nth roots have their principal values. Then

$$\Delta=\zeta^{-g_1}\Omega(a,b;\lambda,\mu,\nu;\zeta^{g_0})/\sqrt[n]{\varphi_0(\mu-\nu)}$$

with g_0 and g_1 depending on a, b. Hence there exists an infinite subsequence E_4 of E_3 such that, if a_1, b_1 is a fixed and a, b a variable pair in E_4, then a_1/b_1 and a/b lie on the same side of α and are near α; furthermore, the quotient $\Gamma=\Delta/\Delta_1$ of the corresponding numbers Δ and Δ_1 takes the form

$$\Gamma=\Omega(a,b;\lambda,\mu,\nu;\zeta^{g_0})/\Omega(a_1,b_1;\lambda,\mu,\nu;\zeta^{g_0}),$$

g_0 fixed. Choose for M the smallest subfield of H, say of degree m, which contains Γ for some infinite subset E_5 of E_4. Then all m conjugates $\Gamma^{(t)}$ of Γ over R can be written as

$$\Gamma^{(t)}=\Omega(a,b;S\lambda,S\mu,S\nu;\zeta^g)/\Omega(a_1,b_1,S\lambda,S\mu,S\nu;\zeta^g),$$

where g runs over $0,\cdots,n-1$, and S over the automorphisms which change K into the conjugate fields; we can write, instead

$$\Gamma^{(t)}=\Omega(a,b;\lambda_t,\mu_t,\nu_t;\zeta^{g_t})/\Omega(a_1,b_1;\lambda_t,\mu_t,\nu_t;\zeta^{g_t}),$$

$t=1,\cdots,m$, and choose the notation so that $\lambda_t=\lambda_\tau$, $\mu_t=\mu_\tau$, $\nu_t=\nu_\tau$, $g_t=g_\tau$ or $\lambda_t=\mu_\tau$, $\mu_t=\lambda_\tau$, $\nu_t=\nu_\tau$, $g_t=g_\tau$ only if $t=\tau$. Of the fields conjugate to M, let r_1 be real and $2r_2$ complex; hence $r_1+2r_2=m$. One can show that, if θ was chosen suitably, then $r_2\geqq k^2+h+2$; hence, by Dirichlet's theorem, the rank r of the group of units in M is

$$r=r_1+r_2-1=m-r_2-1\leqq m-k^2-h-3.$$

By construction, the principal ideal (Γ) is of the form

$$(\Gamma)=(\Delta/\Delta_1)=(\mathfrak{A}/\mathfrak{A}_1)\mathfrak{P}_1^{l_1-l_1'}\cdots\mathfrak{P}_j^{l_j-l_j'}.$$

Hence there is an infinite subsequence E_6 of E_5 in which $\Gamma=A_0\Gamma^*$, where A_0 is a fixed number in M and Γ^* belongs to a multiplicative group in M of rank

$$\rho=r+j\leqq r+h\leqq m-k^2-3.$$

Let $P=(X_1,\cdots,X_m)$ be the point in m-dimensional space S of coordinates $H_1=\Gamma^{(1)}/A_0^{(1)},\cdots,H_m=\Gamma^{(m)}/A_0^{(m)}$; here the upper indices denote the conjugates with respect to M. The points belong to a multiplicative group of rank ρ and lie on the algebraic variety in S of dimension $\sigma=2$ defined by

$$H_t=\omega_t\Omega(x,y;\lambda_t,\mu_t,\nu_t;\zeta^{g_t}),\qquad t=1,\cdots,m,$$

where the ω_t are constants. A general theorem of the author [Ann. Mat. Pura Appl. (4) 17, 127–168 (1938), theorem 2.4] implies, therefore, the existence of an infinite subsequence E_7 of E_6 for which the points P satisfy a system of homogeneous multiplicative relations of the form

$$\prod_{t=1}^{m}\Omega(a,b;\lambda_t,\mu_t,\nu_t;\zeta^{g_t})^{C_{tq}}=\text{constant},\quad \sum_{t=1}^{m}C_{tq}=0,\quad q=1,\cdots,\kappa,$$

where the matrix of the integral exponents C_{tq} is at least of rank $m-\rho-\sigma\geqq k^2+1$. On putting $c=a/b$, by homogeneity for the pairs in E_7,

$$\prod_{t=1}^{m}\Omega(c,1;\lambda_t,\mu_t,\nu_t;\zeta^{g_t})=\text{constant},\quad q=1,\cdots,\kappa.$$

Now c tends to α and so the last equations imply the system of identities

$$\prod_{t=1}^{m}\Omega(z,1;\lambda_t,\mu_t,\nu_t;\zeta^{g_t})=\text{constant},\quad q=1,\cdots,\kappa.$$

But a direct discussion shows that at most k^2 independent equations of this type can hold, whence a contradiction. *K. Mahler* (Manchester).

Williams, Christine S., and Pall, G. The thirty-nine systems of quaternions with a positive norm-form and satisfactory factorability. Duke Math. J. 12, 527–539 (1945). [MF 13522]

This note tabulates characteristic properties of the 39 systems of integral generalized quaternions with positive definite norm-forms in which factorization is always possible and unique under conditions much like those for integral Hamiltonian quaternions. A companion paper by Pall in which the asserted properties of these quaternion systems

are proved is to appear elsewhere. The first table exhibits the characterizing symmetric matrix $\mathfrak{a}=[a_{\alpha\beta}]$, its adjoint $[A_{\alpha\beta}]$, Hasse's rational (generic) invariants C_p of $f=x'\mathfrak{a}x$, the values of $\epsilon_\alpha=2(j_\alpha-i_\alpha)$, and the units in terms of integral basis $1, j_1, j_2, j_3$. Here the multiplication table of the basis $1, i_1, i_2, i_3$ is given by

$$i_\alpha^2=-A_{\alpha\alpha}\quad(\alpha=1,2,3),\quad i_\alpha i_\beta=-A_{\alpha\beta}+\sum_{\delta=1}^{3}a_{\gamma\delta}i_\delta,$$

$$i_\beta i_\alpha=-A_{\beta\alpha}-\sum_{\delta=1}^{3}a_{\gamma\delta}i_\delta.$$

In the second table the matrix $[a_{\alpha\beta}]$ is replaced by an equivalent diagonal matrix. The resultant simplification of the multiplication table is a little offset by the need of integrality conditions for the coefficients. However, these conditions are given in the table and are quite simple. Only 5 of the 39 systems are maximal. All other systems may be obtained from the maximal systems by integral transformations. The relationships of the various systems to one another are described in a "genealogical" chart.

A. E. Ross (St. Louis, Mo.).

Skolem, Th. Verallgemeinerungen der Betti-Guidiceschen Formel. Avh. Norske Vid. Akad. Oslo. I. 1940, no. 1, 18 pp. (1940). [MF 14161]

Let K be an algebraic number field, R its ring of integers. For elements $a_1, \cdots, a_n$ from R, define their g.c.d. as follows: let $\mathfrak{a}=(a_1, \cdots, a_n)$ and let h be the class number so that $\mathfrak{a}^h=(a)$, a principal ideal in R; then the g.c.d. of $a_1, \cdots, a_n$ is the algebraic integer $a^{1/h}$, which may not be in R. Let Kd designate the set of all products xd, where $x\epsilon K$. Then the general solution of a system

$$\sum_{i=1}^{n}a_{ij}x_i=0,\qquad j=1,\cdots,m,$$

all a_{ij} in R, $m<n$, in numbers $x_i\epsilon R$ can be written in the form

$$x_i=\sum\pm A_{i_1,\ldots,i_m}u_{i,i_1,\ldots,i_m}/d.$$

The sum is over $i_1, \cdots, i_m$ different from i, and $1\leqq i_1<\cdots<i_m\leqq n$; $A_{i_1,\ldots,i_m}$ designates the determinant formed from columns $i_1, \cdots, i_m$ of the matrix of coefficients of the system; d is the g.c.d. of these determinants; $u_{i,i_1,\ldots,i_m}$ are parameters whose values range over the algebraic integers belonging to the set Kd, and two parameters whose index numbers are the same except for order are identical. The sign is + or − according as the permutation

$$\begin{bmatrix}1, 2, \cdots, & n\\ i, i_1, \cdots, i_m, & \cdots\end{bmatrix}$$

is even or odd, the integers to the right of i_m in the lower row being in their natural order. This is the author's most general theorem for homogeneous systems. Nonhomogeneous systems are also discussed.

I. Niven.

Bergström, Harald. Die Klassenzahlformel für reelle quadratische Zahlkörper mit zusammengesetzter Diskriminante als Produkt verallgemeinerter Gaussscher Summen. J. Reine Angew. Math. 186, 91–115 (1944). [MF 13407]

The actual computation of the class number h of a real quadratic field on the basis of the explicit formula leads to considerable difficulties. The case where the discriminant d is a prime number has been treated by H. Hasse [Math. Z. 46, 303–314 (1940); these Rev. 2, 39]. Hasse developed a product formula for generalized Gaussian sums. By means of this formula, he obtained explicit expressions for the rational numbers U and V in $\epsilon^h=(U+V\sqrt{d})/2$, where ϵ is the fundamental unit of the quadratic field. The author generalizes these results to the case of a composite number d, and, starting from the explicit formula for the class number, he obtains a rational method for the computation of h. Again, a product formula for Gaussian sums is used which is a generalization of Hasse's formula. Let $\mathfrak{G}$ denote the group of relatively prime residues (mod d), let $\mathfrak{H}$ be a subgroup of index k in $\mathfrak{G}$ and let $\mathfrak{J}$ be a subgroup of index l of $\mathfrak{H}$. Every character ω of $\mathfrak{J}$ determines a generalized Gaussian sum $\tau_a(\omega, d)=\sum_x\omega(x)e^{2\pi iax/d}$, where the summation index x ranges over all elements of $\mathfrak{J}$ and where the parameter a is an integer. If $\mathfrak{R}$ is a residue system for $\mathfrak{G}$ modulo $\mathfrak{J}$, then $\mathfrak{R}$ can be decomposed into k subsystems $\mathfrak{r}_i$ corresponding to the k classes into which $\mathfrak{G}/\mathfrak{J}$ breaks up modulo $\mathfrak{H}/\mathfrak{J}$. The product formula deals with the product of the sums $\tau_a(\omega, d)$ where a ranges over the elements of a fixed system $\mathfrak{r}_i$. This product is expressed as a linear combination of Gaussian sums $\tau_t(\chi, d)=\sum_y\chi(y)e^{2\pi iyt/d}$, where χ is a character of $\mathfrak{G}$, y ranges over the elements of $\mathfrak{G}$ and t is a divisor of d.

R. Brauer (Toronto, Ont.).

ANALYSIS

Zygmund, Antoni. A theorem on fractional derivatives. Duke Math. J. 12, 455–464 (1945). [MF 13514]

If

$$(*)\qquad f(x+t)=\sum_{\nu=0}^{k}(\alpha_\nu/\nu!)t^\nu+\epsilon_k(x,t)t^k,$$

where $\epsilon_k(x,t)\to 0$ as $t\to 0$, α_k is called the kth generalized derivative $f_{(k)}(x)$ of f at the point x; and then $\alpha_\nu=f_{(\nu)}(x)$ for $\nu=1, \cdots, k-1$. If

$$(\ddagger)\qquad I_\sigma(f)=(1/\Gamma(\sigma))\int_0^x(x-t)^{\sigma-1}f(t)dt,\quad \sigma>0, x>0,$$

and $\alpha>0$ is fractional, $f_{(\alpha)}(x)$ is defined as $g_{([\alpha]+1)}(x)$, where $g=I_{[\alpha]+1-\alpha}(f)$. If $0<\alpha'<\alpha$ and $f_{(\alpha)}(x)$ exists, then so does $f_{(\alpha')}(x)$. If $0<\alpha\leqq 1$ then $f_{(\alpha)}(x)=f^{(\alpha)}(x)$.

Weyl has shown that if $f(x)\epsilon\mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ then $f_{(\beta)}(x)$ exists and is continuous if $0<\beta<\alpha$. This is false if $\beta=\alpha$. Here the author proves that, if $f_{(k)}(x)$ exists in a set E of positive measure, where k is an integer, and if, for some $0<\alpha<1$, $|\epsilon_k(x,t)|\leqq A_x|t|^\alpha$ for $|t|<\delta_x$, then $f_{(k+\beta)}(x)$ exists for almost all $x\epsilon E$ if $0<\beta<\alpha$. [The last condition is incorrectly printed in the paper as $0<\alpha<\beta$.] The main step in the proof is to show that $f(x)=r(x)+s(x)$, where $r^{(k)}(x)$ exists for all x and satisfies Lip α, while $s(x)=0$ in a perfect set $P\subset E$ of measure nearly that of E and $|s(x)|$ is not too big in the shorter intervals contiguous to P. This is an extension of a result of Marcinkiewicz; a detailed proof is given. It is then easily shown by means of Weyl's theorem that $r_{(k+\beta)}(x)$ exists for all x and, by means of a lemma quoted from Marcinkiewicz, that $s_{(k+\beta)}(x)$, and indeed $s_{(k+\alpha)}(x)$, exist for almost all $x\epsilon P$.

L. S. Bosanquet.

Roussel, André. Sur l'approximation locale des fonctions continues. Bull. Soc. Math. France 69, 97–132 (1941). [MF 13241]

Let $f(x)$ be continuous and denote the increment $f(x+h)-f(x)$ by Δf. The problem studied is to write Δf

in the form $g(x, h)+\epsilon\phi(h)$, where $g(x, h)$ vanishes with h, and $\phi(h)$ is a preassigned function decreasing to zero. If $\phi(h)=h$ and $g(x, h)=f'(x)h$, this reduces to the familiar differential. A variety of results of this character are obtained by use of the Weierstrass-Gauss and Poisson kernels.
H. Pollard (New Haven, Conn.).

Chlodovsky, I. The differential properties of functions with one non-negative finite difference of order n. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 620–622 (1945). [MF 14415]

If $f(x)$ is bounded in an interval and has its differences of order n ($n\geqq 2$) nonnegative there, then $f^{(n-2)}(x)$ exists and is continuous and convex. The author was evidently not aware that this had been proved before by Popoviciu [Mathematica, Cluj 8, 1–85 (1934)] and by Boas and Widder [Duke Math. J. 7, 496–503 (1940); these Rev. 2, 219]. The author's method is much the same as that of Boas and Widder. *R. P. Boas, Jr.* (Providence, R. I.).

Theory of Functions of Complex Variables

✶Markuševič, A. I. Èlementy Teorii Analitičeskih Funkcii. [Elements of the Theory of Analytic Functions]. UČPEDGIZ, Moscow, 1944. 544 pp. (Russian)

Elements of the theory of analytic functions of a complex variable are developed with many examples and exercises. The book is essentially an introduction to the general theory and a preparation for the study of special functions and of the various recent developments which did not come within the scope of the book. After a descriptive and historical introduction, in chapter I the author discusses fundamental notions of the theory: complex numbers and operations on them, point sets in the plane, complex functions, differentiation, integration, the Cauchy-Goursat theorem and primitive functions. Chapter II is concerned with power series, operations on series, singularities, analytic continuation, and the principle of the maximum. Ramifications of Cauchy's theorem are dealt with in chapter III. In particular, Morera's theorem, Laurent series, the monodromy theorem, the theorem of Rouché and harmonic functions are treated. Elementary functions, including rational, trigonometric, exponential and logarithmic functions, occupy the author's attention in chapter IV, while in chapter V he discusses conformal mappings, including stereographic projections, maps by linear functions and applications to cartography. *E. F. Beckenbach* (Los Angeles, Calif.).

Wintner, Aurel. A property of the elliptic modular net. Duke Math. J. 12, 451–454 (1945). [MF 13513]

The discriminant of the cubic polynomial

$$(dp/dw)^2=4p^3-g_2p-g_3$$

in $p(w, \omega_1, \omega_2)$ is $\Delta(\omega_1, \omega_2)=g_2^3-27g_3^2=\omega_1^{-12}\Delta(1, \omega)$, where $m_1, m_2=0, \pm 1, \pm 2, \cdots$, but $(m_1, m_2)\neq(0, 0)$,

$$g_2(\omega_1, \omega_2)=60\sum{}'(m_1\omega_1+m_2\omega_2)^{-4},$$
$$g_3(\omega_1, \omega_2)=140\sum{}'(m_1\omega_1+m_2\omega_2)^{-6},$$

and $\omega_2/\omega_1=\omega=x+iy$, $y>0$. The function $\Delta(\omega)=\Delta(1, \omega)$ is regular in the half plane $y>0$ and is relatively invariant under every substitution of the modular group. The x-axis is a natural boundary of $\Delta(\omega)$. The Eisenstein series g_2, g_3 exhibit for $\Delta(\omega)$ a formal pole at every rational x. It turns out that, if a certain x-set Z of measure 0 is discarded, $\Delta(\omega)=\Delta(x+iy)$ tends to a finite nonvanishing limit $\Delta(x)$ as $y\to 0$. The exclusion of a zero set Z is essential. The boundary function $\Delta(x)$ proves to exist, for almost all x, not only in the radial sense, but also as a Stolzian limit; that is, if the real number x does not belong to a certain set of measure 0, the function $\Delta(\omega)$ ($y>0$) tends to a finite limit $\Delta(x)$ as ω tends to x within any fixed wedge $\epsilon<\arg(\omega-x)<\pi-\epsilon$, $\epsilon>0$. The proof is based on an Eulerian factorization

$$\Delta(\omega_1, \omega_2)=(2\pi/\omega_1)^{12}q^2\prod_{n=1}^{\infty}(1-q^{2n})^{24},$$

where $q=e^{i\omega\pi}$, $|q|<1$. *S. C. van Veen* (Dordrecht).

Dalzell, D. P. Note on theta-Fuchsian functions. J. London Math. Soc. 19, 135–137 (1944). [MF 13633]

Let $\theta(m, z)$ be a theta-Fuchsian function of rank m (in other terminology, an automorphic form of weight or dimension $-2m$, belonging to a group with the unit circle as principal circle); then the number of zeros $N(0)$ and the number of poles $N(\infty)$ in the fundamental region have the difference

$$N(0)-N(\infty)=2m\pi^{-1}A(R), \tag{1}$$

with $A(R)=\iint_R(1-|z|^2)^{-2}dxdy$, the non-Euclidean area of R. The author gives a new proof of (1) by showing that the nonanalytic function $F(z)=\theta'(m, z)/\theta(m, z)-2m\bar{z}/(1-|z|^2)$ possesses the transformation property of a theta-Fuchsian function of rank one. The result then follows by integrating $F(z)$ around the contour of R. *H. Rademacher.*

Fedoroff, V. S. Sur la monogénéité dans l'espace. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 222–223 (1945). [MF 12945]

Let u, v, P and Q be real-valued functions of x, y and z having continuous second partial derivatives in a region D. Let (1) $u+iv\sim\mathbf{a}$ mean that $\nabla u=\nabla v\times\mathbf{a}$, $\nabla v=\mathbf{a}\times\nabla u$ and $(\nabla u)^2\neq 0$, where $\mathbf{a}$ is a vector field over D. The author announces four results, of which the following two are the simplest to state. (a) Given (1) then $f(u+iv)\sim\mathbf{a}$ and $f(u-iv)\sim-\mathbf{a}$ for every function f which is holomorphic and has nonzero derivative for the values of $u+iv$ in D. (b) If $u+iv\sim\mathbf{a}$ and $P+iQ\sim\mathbf{a}$ in D then there is a neighborhood of each point in D in which $P+iQ$ is a holomorphic function of $u+iv$, with nonzero derivative.
L. H. Loomis (Cambridge, Mass.).

Picone, Mauro. Osservazione alla nota di Maria Zevi. Pont. Acad. Sci. Acta 5, 155–158 (1941). (Italian. Latin summary) [MF 11919]

Special sufficient conditions for the analyticity of a function of a complex variable are established and discussed. [The note quoted in the title appeared in the same Acta 5, 143–152 (1941).] *M. H. Heins* (Providence, R. I.).

Ferrand, Jacqueline. Fonctions préharmoniques et fonctions préholomorphes. Bull. Sci. Math. (2) 68, 152–180 (1944). [MF 13249]

A complex function $f(z)=P(z)+iQ(z)$, defined at the vertices of a bounded network r of square meshes, remains continuous on the sides of the network if defined there by linear extrapolation; $f(z)$ is said to be preholomorphic on r provided that for each mesh q of r we have (1) $\int_q f(z)dz=0$. If a, b, c, d denote the vertices of q taken in positive order, the condition (1) is equivalent to the conditions

$$Q(d)-Q(b)=P(c)-P(a),\quad Q(c)-Q(a)=P(b)-P(d),$$

which prefigure the Cauchy-Riemann partial differential conditions.

Those vertices of r which can be connected with a given fixed vertex ω of r by a continuous succession of diagonals of meshes of r are called even vertices of r, the others odd. It is shown that, for simply-connected domains, if the values of $P(z)$ are given on the set E of even vertices of r, then the values of $Q(z)$ exist and are uniquely determined to within an arbitrary additive constant on the odd vertices of r if and only if $P(z)$ is preharmonic on E; that is, if and only if for each interior vertex z_0 of E the value of $P(z_0)$ is the average of the values of $P(z)$ at the four adjacent vertices of E.

Several properties of preholomorphic functions are developed: the modulus of such a function attains its maximum on the frontier, the Cauchy integral theorem holds, and preholomorphic primitive and derivative functions exist. By means of preholomorphic functions, a new proof is given of the result that, under the continued quartering of meshes, the limit of a convergent sequence of preharmonic functions coincides on its dense set of definition with a harmonic function. An application of preholomorphic functions is given in proving the Riemann conformal mapping theorem.

E. F. Beckenbach (Los Angeles, Calif.).

Ferrand, Jacqueline. Nouvelle démonstration d'un théorème de M. Ostrowski. C. R. Acad. Sci. Paris **220**, 550–551 (1945). [MF 14091]

The author gives a proof of a theorem of Ostrowski concerning the conformal mapping of simply-connected regions [Prace Mat.-Fiz. **44**, 371–471 (1936), in particular, §29] with the aid of the theory of normal families.

M. H. Heins (Providence, R. I.).

Macintyre, A. J., and Rogosinski, W. W. Some elementary inequalities in function theory. Edinburgh Math. Notes no. 35, 1–3 (1945). [MF 14444]

Let $f(z)$ be regular in $|z|\leqq 1$. The inequality

$$(1-|z|^2)|f(z)|\leqq(2\pi)^{-1}\int_0^{2\pi}|f(e^{i\theta})|\,d\theta$$

due to Egerváry [Math. Ann. **99**, 542–561 (1928)] and the inequality

$$\frac{(1-|z|^2)^2}{|z|+(1+|z|^2)^{\frac12}}|f'(z)|\leqq(2\pi)^{-1}\int_0^{2\pi}|f(e^{i\theta})|\,d\theta$$

are proved by elementary methods. Each inequality is the best possible.

P. Civin (Buffalo, N. Y.).

Bermant, A. Sur la variation de la dilation d'une fonction régulière. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 271–273 (1944). [MF 12665]

Let $f(z)$ be regular in $|z|<1$ with $f(0)=0$, and let D^* be the "star" of $f(z)$ with respect to the point $w=f(0)=0$: D^* is defined to be the largest star-like domain containing the point 0 and lying in the domain of values of $f(z)$, $|z|<1$. The author proves the following result. Let λ^* be the frontier of D^* and suppose that λ^* is given by $w=\rho(\theta)e^{i\theta}$, where $1-\epsilon<\rho(\theta)<1+\epsilon$ and where, except on the radial segments belonging to λ^*, $\rho(\theta)$ satisfies $|\rho(\theta+\Delta\theta)-\rho(\theta)|<K\epsilon|\Delta\theta|$. Then

$$|f'(0)|=(S^*/\pi)^{\frac12}+O\{(n+1)\epsilon^2|\log\epsilon|\}.$$

Here S^* is the area of D^* and n is the number of radial segments belonging to λ^*. This result is obtained by combining a theorem of A. Marchenko [same C. R. **1935 I**, 287–290 (1935)] with the inequality $f'(0)\leqq(S^*/\pi)^{\frac12}$ [G. M. Golusin, Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 617–619 (1937); A. Bermant, C. R. Acad. Sci. Paris **207**, 31–33 (1938)]. A sharpened form of the inequality has been given by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **26**, 616–621 (1940); these Rev. **2**, 79].

D. C. Spencer.

Bermant, A. Dilatation d'une fonction modulaire et problèmes de recouvrement. Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 285–324 (1944). (Russian. French summary) [MF 12288]

Let R denote the class of functions $w=f(z)$ which are regular in the unit circle $|z|<1$ and normalized by the conditions that $f(0)=0$, $|f'(0)|=1$. Using the principle of subordination (Lindelöf principle) with the modular functions as superordinate [J. E. Littlewood, Lectures on the Theory of Functions, Oxford University Press, 1944, chapter 2; these Rev. **6**, 261], the author proves some results concerning the covering of the w-plane by the values of any function f of R. The paper is divided into two parts. In the first part a study is made of the behavior of a modular function regular in $|z|<1$, which omits the values w_1, w_2 (and of course $w=\infty$), as the exceptional points w_1, w_2 vary. The theorems of the second part are based on this behavior. Among others, the following two theorems are proved. (1) Any function of R covers at least one of the two radii of each diameter of the circle $|w|<A_1$, where $A_1=0.228\cdots$. The value A_1 cannot be replaced by any smaller one. (2) Any function of R covers a segment containing $w=0$ of each line passing through $w=0$ and the length of the segment is at least equal to $2A_1=0.456\cdots$. The value $2A_1$ is best possible. These results are extensions of a theorem of Szegö.

D. C. Spencer (Stanford University, Calif.).

Robertson, M. S. The coefficients of univalent functions. Bull. Amer. Math. Soc. **51**, 733–738 (1945). [MF 13612]

Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$, where the c_n are real, be regular and convex in the direction of the imaginary axis for $|z|<1$. As shown by the author [Ann. of Math. (2) **37**, 374–408 (1936)] this implies $|c_n|\leqq|c_1|$. He now adds the inequality

$$c_{n-1}-c_{n+1}\leqq 4n(n^2-1)^{-1}(c_1-|c_n|)$$

if $c_1>0$. If n is even, the left side can be replaced by $|c_{n-1}-c_{n+1}|$. These estimates are best possible.

W. W. Rogosinski (Newcastle-upon-Tyne).

Rosenblatt, Alfred. On power series in the unit circle. Actas Acad. Ci. Lima **6**, 39–42 (1943). (Spanish) [MF 14279]

A more detailed presentation appeared in Revista Ci., Lima **45**, 195–225 (1943); these Rev. **5**, 176.

Dufresnoy, Jacques. Sur quelques progrès récents de la théorie des fonctions d'une variable complexe. Revue Sci. (Rev. Rose Illus.) **79**, 608–612 (1941). [MF 13820]

A brief account is given of the Nevanlinna theory of meromorphic functions and some of its recent developments due to the author.

M. H. Heins (Providence, R. I.).

Keldych, M. Sur l'approximation des fonctions holomorphes par les fonctions entières. C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 239–241 (1945). [MF 14022]

The author states a number of theorems, mostly on the closeness of the possible approximation. For example, if $f(z)$ is analytic in $|\arg z|<\frac12\alpha$ and continuous in the

closed angle, an entire function $G(z)$ exists for which $|f(z)-G(z)|<\epsilon \exp(-|z|^{(\pi/\alpha)-\eta})$, $\epsilon>0$, $\eta>0$. A similar result is given for a strip. Under additional restrictions on $f(z)$, the order of $G(z)$ can be specified. [These results overlap somewhat those of Kober, Trans. Amer. Math. Soc. **56**, 7–31 (1944); these Rev. **5**, 258.] That any continuous function can be uniformly approximated on the real axis by an entire function was first shown by Carleman [Ark. Mat. Astr. Fys. **20B**, no. 4 (1927); see also A. Roth, Comment. Math. Helv. **11**, 77–125 (1938); for L^p approximation, Kober, Trans. Amer. Math. Soc. **54**, 70–82 (1943); these Rev. **4**, 271]. The author states that, if $\mu(t)$ is the maximum of $|f'(x)|$ for $|x|<t$ and $\limsup_{t\to\infty}\{\log \mu(t)\}/\log t=\nu$, then there is an entire function $G(x)$ of order not exceeding $\nu+1$, such that $|f(x)-G(x)|<\epsilon$ on $(-\infty, \infty)$. The results quoted are essentially "best possible." *R. P. Boas, Jr.*

Thullen, Peter. On the theory of analytic functions of several complex variables. Domains of regularity and Reinhardt's domains of meromorphy. Revista Unión Mat. Argentina **11**, 33–46 (1945). (Spanish. French summary) [MF 14664]

Cartan and Thullen have shown that a domain of regularity is also a domain of meromorphy [Math. Annalen **106**, 617–647 (1932)]. The author proves that, if a Reinhardt domain is a domain of meromorphy, then it is a domain of regularity. Having this result he then proves the following result: pseudo-convexity is a necessary and sufficient condition for a (proper or improper) Reinhardt domain to be a domain of regularity. This solves Levi's problem completely for Reinhardt domains. Previously the solution for Reinhardt domains was known under certain restrictive conditions on the boundary. Part of the results contained in the present paper were announced by the author in 1934 [C. R. Acad. Sci. Paris **199**, 1016–1018 (1934)].
W. T. Martin (Syracuse, N. Y.).

Lelong, Pierre. Sur quelques problèmes de la théorie des fonctions de deux variables complexes. Ann. Sci. École Norm. Sup. (3) **58**, 83–177 (1941). [MF 12896]

Let $f(x, y)$ be an analytic function of two complex variables in a domain which contains in its interior a plane domain $x\epsilon d$, $y=y_0$. Then f can be developed in a Hartogs series

$$f(x, y)=\sum_{n=0}^{\infty} A_n(x)(y-y_0)^n \tag{1}$$

which converges for $x\epsilon d$ and y in the neighborhood $N(x)$ of y_0. The author studies the relation between f and the sequence of subharmonic functions $U_n(x)=n^{-1}\log|A_n(x)|$. Following Hartogs he considers the two functions

$$U(x)=\limsup_{n\to\infty} U_n(x), \quad V(x)=\limsup_{x'\to x} U(x'),$$

the first of which is in general discontinuous and the second of which is a (semicontinuous) subharmonic function. Hartogs showed that the series (1) converges uniformly in every closed domain interior to

$$[x\epsilon d,\ |y-y_0|<\exp\{-N(x)\}].$$

One of the theorems of chapter 1 is the following. Given a function $f(x, y)$ holomorphic in a neighborhood D of the closed plane domain $x\epsilon d$, $y=y_0$, if one denotes by $\nu(n)$ the number of points of intersection of the variety $\varphi_n(x, y)=0$ with the plane $y=y_0$ which are located on d, in order that there exists a singularity of $f(x, y)$ located at a finite distance which has a projection on the plane $y=y_0$ in the interior of d, one must have $\liminf \nu(n)/n<A$, A being a constant which depends essentially on the distance of the set S from the domain d taken in a direction parallel to the plane $x=$constant. If $\lim \nu(n)/n=\infty$, the function $f(x, y)$ has no singularity at finite distance over the domain d. In this theorem $\varphi_n(x, y)$ denotes the partial derivative $\partial^n f(x, y)/\partial y^n$. The points x where $U(x)<V(x)$ are of importance throughout the chapter.

In chapter 2 the author investigates the singularities of f at a finite distance. The first result obtained states that, given any domain H in E_4 defined by

$$[x\epsilon d,\ |y-y_0|<R(x)], \tag{2}$$

where $V_1(x)\equiv -\log R(x)$ is any subharmonic function, it is possible to construct a function $f(x, y)$ whose Hartogs development (1) has H as its domain of uniform convergence; that is, the series (1) converges uniformly in every closed subdomain of H. The development so obtained is lacunary (but may be otherwise if desired) and the function $f(x, y)$ cannot be continued analytically beyond H. This result shows that the domain (2) is a domain of regularity whenever $-\log R(x)$ is subharmonic. Previously this had been proved only under the additional assumption that $R(x)$ has continuous second order partial derivatives (as a function of two real variables).

In chapter 3 the author studies the case in which f has no singularities in the domain $x\epsilon d$, $|y|<\infty$. In place of the singularities of f he now studies the growth of the entire functions obtained by holding x constant. This study is related to other questions of interest. *W. T. Martin.*

Martinelli, Enzo. Sulla formula di Cauchy n-dimensionale e sopra un teorema di Hartogs nella teoria delle funzioni di n variabili complesse. Comment. Math. Helv. **17**, 201–208 (1945).

Martinelli, Enzo. Formula di Cauchy $(n+1)$-dimensionale per le funzioni analitiche di n variabili complesse. Comment. Math. Helv. **18**, 30–41 (1945).

In früheren Arbeiten des gleichen Verfassers werden im Raume S_{2n} der n komplexen Veränderlichen $z_1, \cdots, z_n$ gewisse Verallgemeinerungen der klassischen Cauchyschen Integralformel für Integrationsflächen von $2n-1$ bzw. n Dimensionen untersucht. Auf Grund der beiden aufgestellten Integralformeln gab der Verfasser je einen neuen Beweis eines grundlegenden Hartogs'schen Satzes an; im ersten Falle in allgemeiner Form, im zweiten sich auf $n=2$ und konvexe Bereiche beschränkend [siehe Comment. Math. Helv. **15**, 340–349 (1943); diese Rev. **6**, 61].

In der ersten der vorliegenden Arbeiten wird nunmehr mittels der "n-dimensionalen" Cauchyschen Integralformel der Beweis des Hartogs'schen Satzes für jedes beliebige n und ohne die einschränkende Voraussetzung der Konvexität des zu Grunde liegenden Bereiches erbracht. In der zweiten Arbeit weist der Verfasser darauf hin, dass für jede der Dimensionen $n, n+1, \cdots, 2n-1$ eine Cauchysche Integralformel aufgestellt werden kann. Der Beweis wird für $n+1$ durchgeführt, indem nachgewiesen wird, dass die Gültigkeit der Integralformel nur durch gewisse topologische Eigenschaften der Integrationsfläche bedingt ist. Hiernach können also die Werte einer in einem Bereiche R_{2n} analytischen Funktion $f(z_1, \cdots, z_n)$ mittels eines $(n+1)$-fachen Integrals eindeutig angegeben werden, wenn nur die Integrationsfläche den angegebenen topologischen Bedingungen genügt. *P. Thullen* (Quito).

Severi, Francesco. Le funzioni periodiche di più variabili. Comment. Math. Helv. 18, 16–29 (1945).

The author sketches a theory of functions of r complex variables with $\nu<2r$ periods which are meromorphic at finite distance. Such functions arose first in the work of Weierstrass and Painlevé on functions admitting an addition theorem. Examples of such quasi-Abelian functions can be constructed by considering certain neutral linear series on algebraic curves. The statements of this paper are largely based on such special properties. Thus certain results on the structure of quasi-Abelian varieties V_r as products of Abelian varieties and linear spaces are stated. In particular, the absolute transitivity of the group of transformations in a V_r induced by the addition of residues modulo the periods is a necessary and sufficient condition that V_r is an Abelian variety. *O. F. G. Schilling.*

Theory of Series

Dalzell, D. P. On $\frac{22}{7}$. J. London Math. Soc. 19, 133–134 (1944). [MF 13632]

By the use of integral calculus the author establishes the inequalities $\frac{22}{7}-\frac{1}{1260}>\pi>\frac{22}{7}-\frac{1}{630}$. He then proceeds to develop a series $\pi=\frac{22}{7}+\sum_{n=1}^{\infty} a_n$, where the a_n's are less in magnitude than the terms of a geometric series of ratio $\frac{1}{1024}$. *T. Fort* (Athens, Ga.).

Claudian, Virgil. Einige neue Betrachtungen über die Kriterien erster und zweiter Art für die Reihen mit positiven Gliedern. Bull. École Polytech. Bucarest [Bul. Politehn. București] 13, 31–41 (1942). [MF 13564]

The author proves three theorems of which the following is the first and simplest. If $a_n\geqq 0$ and $n(a_n/a_{n+1}-1)\rightarrow l$ as $n\rightarrow\infty$, then $\log(1/a_n)/\log n\rightarrow l$. The other two theorems are similar, employing expressions occurring in well-known extensions of Raabe's test for convergence. *T. Fort.*

Bohr, Harald. On general convergence criteria for series of positive terms. Mat. Tidsskr. B. 1945, 1–9 (1945). (Danish) [MF 14257]

The author reports the content of one of his unpublished manuscripts giving a new approach to criteria of the general type given by Pringsheim [Math. Ann. 35, 297–394 (1890); in particular, 343 ff.]. *František Wolf* (Berkeley, Calif.).

Wachs, S. Sur un problème de théorie des fonctions solidaire du théorème de Fermat. Rev. Sci. (Rev. Rose Illus.) 78, 297–298 (1940). [MF 13318]

As the author has remarked elsewhere, to prove Fermat's theorem is equivalent to proving that, if a, b, c are integers, then no one of the coefficients in the power series expansion in terms of z of the function $(d/dz)\log(1-az)(1-bz)(1-cz)$ can be zero [C. R. Acad. Sci. Paris 211, 55–57 (1940); these Rev. 3, 67]. In this paper he shows that, if a, b, c are restricted to be real (rather than integral), then at most one of the coefficients in the power series expansion can be zero. *H. W. Brinkmann* (Swarthmore, Pa.).

Bosanquet, L. S. Note on the converse of Abel's theorem. J. London Math. Soc. 19, 161–168 (1944). [MF 13638]

Generalizing results of Hardy, Littlewood and others the author considers

$$f(s)=\sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$$

and proves that if

$$\lim_{\delta\rightarrow+0}\ \limsup_{n\rightarrow\infty}\ \max_{\lambda_n<\lambda_m<(1+\delta)\lambda_n} |a_{n+1}+a_{n+2}+\cdots+a_m|=0$$

and if $f(s)\rightarrow S$ as $s\rightarrow+0$ then $\sum a_n$ converges to S. *N. Levinson* (Cambridge, Mass.).

Szász, Otto. On some summability methods with triangular matrix. Ann. of Math. (2) 46, 567–577 (1945). [MF 14120]

The author considers the sequence to sequence transformation $t_n=\sum_{k=0}^{n} a_{nk}s_k$ with triangular matrix (a_{nk}). A necessary and sufficient condition that $\lim t_n$ exists when $(C, r)\lim s_n$ exists is established and then applied to three particular triangular transforms; corollaries are obtained in each of these three cases indicating when they are more powerful than any (C, r). *D. Moskovitz.*

Rudberg, Hans. Un rapport entre quelques méthodes de sommation. Ark. Mat. Astr. Fys. 30A, no. 10, 15 pp. (1944). [MF 12006]

For each $r=1, 2, \cdots$, let $A(r)$ denote a matrix of elements $a_{nk}(r)$ for which $a_{nk}(r)>0$ when $0\leqq k\leqq n$, $a_{nk}(r)=0$ when $k>n$, $\lim_{n\rightarrow\infty} a_{nk}=0$ for each k, and $\sum_{k=0}^{n} a_{nk}=1$ for each n. Let $B(0)$ be the identity matrix. For each $r=1, 2, \cdots$, let $b_{nk}(r)$ denote the elements of the product matrix $B(r)$ defined by $B(r)=A(r)A(r-1)\cdots A(1)$. Corresponding to each sequence $r_0, r_1, \cdots$ of nonnegative integers, let $B(r_n)\equiv B(r_1, r_2, \cdots)$ denote the matrix method of summability by means of which a sequence $s_0, s_1, \cdots$ is summable to s if $s_n(r_n)\rightarrow s$ as $n\rightarrow\infty$, where

$$s_n(r_n)=\sum_{k=0}^{n} b_{nk}(r_n)s_k, \qquad n=0, 1, \cdots.$$

It is shown that, if $r_n\rightarrow\infty$ sufficiently slowly, $B(r_n)$ is a regular method of summability which includes all the methods $B(r)$. Moreover, if r_n is monotone increasing, if $r_n'\geqq r_n$, and if $B(r_n')$ and $B(r_n)$ are regular, then $B(r_n')$ includes $B(r_n)$. Applications involving the Cesàro and Euler methods of summability are given. *R. P. Agnew.*

Good, I. J. On the regularity of moment methods of summation. J. London Math. Soc. 19, 141–143 (1944). [MF 13635]

Let $f(t)$ be a nonnegative function for which the moments

$$c_n=\int_0^{\infty} f(t)t^n dt, \qquad n=0, 1, 2, \cdots,$$

exist and are positive. Let t_0, finite or $+\infty$, be the least number such that $f(t)=0$ for almost all $t>t_0$. The function $f(t)$ determines a method for evaluation of series by means of which $u_0+u_1+u_2+\cdots$ is evaluable to L if

$$\sigma(t)=\int_0^t f(x)\sum_{n=0}^{\infty}(u_n/c_n)x^n dx$$

exists when $0\leqq t<t_0$ and $\sigma(t)\rightarrow L$ as $t\rightarrow t_0$. It is shown, by use of the Silverman-Toeplitz conditions, that the method is regular. In case $f(t)=e^{-t}$, the method is the Borel integral method [not the Borel exponential method, as the author says]. *R. P. Agnew* (Ithaca, N. Y.).

Fuchs, W. H. J. A theorem on Hausdorff's methods of summation. Quart. J. Math., Oxford Ser. 16, 64–77 (1945). [MF 14167]

The author treats a fundamental problem involving methods of summability introduced by Hurwitz and Silver-

man [Trans. Amer. Math. Soc. 18, 1–20 (1917)]. Each sequence $\mu_0, \mu_1, \cdots$ of real or complex numbers generates a transformation

$$(H) \qquad t_n = \sum_{k=0}^{n} \binom{n}{k} \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} \mu_{k+p} s_k$$

by means of which a sequence $s_0, s_1, \cdots$ is summable to t if $t_n \to t$ as $n \to \infty$. A method H is conservative if $\lim t_n$ exists whenever $\lim s_n$ exists. The product $H'H''$ of two methods generated by sequences μ_n' and μ_n'' is generated by the sequence $\mu_n'\mu_n''$. If $\mu_n \neq 0$ for each n, then H has an inverse H^{-1} generated by μ_n^{-1}. If $\mu_n' \neq 0$ for each n, then the equation $H'' = H''(H')^{-1}H'$ shows immediately that H'' is at least as strong as H' if and only if $H''(H')^{-1}$ is conservative Thus, in case $\mu_n' \neq 0$ for each n, H'' is at least as strong as H' if and only if there is a conservative Θ of the form H such that $H'' = \Theta H'$.

Since these results were given by Hurwitz and Silverman, several authors have treated the case in which $\mu_n = 0$ for at least one n. Fuchs [Proc. Cambridge Philos. Soc. 40, 188–196 (1944); these Rev. 6, 46] gave an example of methods H' and H'', not both conservative, such that H'' is at least as strong as H' and there is no conservative Θ such that $H'' = \Theta H'$. The paper under review gives the following theorem. Let T' and T'' be conservative methods generated by real sequences μ_n' and μ_n''. Then each real sequence summable T' is also summable T'' if and only if there is a conservative Θ of the form H such that $T'' = \Theta T'$.

A trivial exception mars the theorem. If $\mu_n' = 0$ for all n and if $\mu_0'' = 1$ while $\mu_n' = 0$ when $n > 0$, then T' and T'' both evaluate all sequences and hence are conservative and equally strong; but there is no Θ such that $T'' = \Theta T'$. The proof of the theorem is based on the theorem of Hausdorff [Math. Z. 9, 74–109 (1921)] that H is conservative if and only if the sequence μ_n is the moment sequence $\mu_n = \int_0^1 t^n d\psi(t)$ of a function $\psi(t)$ having bounded variation over $0 \leq t \leq 1$. Properties of moment functions (Mellin transforms) and of linear transformations in linear spaces are used extensively in the proof. *R. P. Agnew* (Ithaca, N. Y.).

Mears, Florence M. Nörlund summability of Cauchy products. Ann. of Math. (2) 46, 563–566 (1945). [MF 14119]

A sequence $p_0, p_1, \cdots$ of real or complex numbers for which $P_n = p_0 + \cdots + p_n \neq 0$ generates a Nörlund method P of summability by which a series $\sum u_n$, with partial sums U_n, is summable to U if $\sigma_n \to U$ as $n \to \infty$, where $\sigma_n = P_n^{-1} \sum_{k=0}^{n} p_{n-k} U_k$. Let Q be a Nörlund method generated by $q_0, q_1, \cdots$; and let R be the Nörlund method generated by $r_n = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0$, $n = 0, 1, \cdots$. Modifications of the following theorem of the author [Bull. Amer. Math. Soc. 41, 875–880 (1935)] are given. Let P and Q be regular Nörlund methods generated by sequences of real nonnegative numbers, let $\sum u_n$ and $\sum v_n$ be summable P and Q to U and V respectively, and let one of the summabilities be absolute. Then the Cauchy product-series

$$\sum w_n \equiv \sum (u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0)$$

is summable R to UV.

In contradiction to a theorem of Silverman and Szász [Ann. of Math. (2) 45, 347–357 (1944); these Rev. 5, 236], it is shown that a false statement is obtained by deleting the hypothesis that P and Q are regular. Correct statements are obtained by deleting the hypothesis that P and Q are regular and adding the hypothesis (1) that $U = V = 0$ or (2) that $p_n/R_n \to 0$ and $q_n/R_n \to 0$ as $n \to \infty$. Three further theorems apply to methods P and Q for which p_n and q_n are complex. *R. P. Agnew* (Ithaca, N. Y.).

Hill, J. D. Summability of sequences of 0's and 1's. Ann. of Math. (2) 46, 556–562 (1945). [MF 14118]

According to the law of large numbers almost all sequences of 0's and 1's are $(C, 1)$ summable to the value $\frac{1}{2}$. This suggests at least two possible directions for generalization: (i) to more general sequences, (ii) to more general methods of summability. In the present paper the author discusses only problem (ii). His method is the usual one of dyadic mapping on (0, 1), together with simple applications of the Rademacher functions and "homogeneous" point sets. [That these devices have already been used to study many similar problems for subseries and subsequences, and problem (i) in particular, seems to have been overlooked. See, for example, Paley and Zygmund [Proc. Cambridge Philos. Soc. 26, 337–357 (1930)] or Buck and Pollard [Bull. Amer. Math. Soc. 49, 924–931 (1943); these Rev. 5, 117].]

Let T be a regular method of summability, with matrix $\{a_{nk}\}$ such that (1) $a_{nk} = o(1)$, $n \to \infty$, $k \geq 0$; (2) $\lim_{n\to\infty} \sum_k a_{nk} = 1$; (3) $\sum_k |a_{nk}| \leq M$. The author proves that (a) almost all or almost none of the 0–1 sequences are summable T; (b) in the former case they are almost all summable to $\frac{1}{2}$; (c) in order that almost all be summable T it is sufficient that $\sum_n (\sum_k a_{nk}^2)^q < \infty$ for some positive q, and necessary that $\sum_k a_{nk}^2 = o(1)$, $n \to \infty$; (d) the set of T-summable sequences is of the first category. Applications are made to special cases.

[The author's proofs depend heavily on condition (3). But if we are dealing with 0–1 sequences alone this condition seems to be out of place, for (1) and (2) are enough to guarantee the "regularity" of T with respect to them. Moreover, Paley and Zygmund in the paper cited dispense with (3). This suggests that, at least for the measure-theoretic portions of the present paper, the extra condition is irrelevant. More refined proofs, of the kind given in the second paper cited, show that (a), (b), (c) are in fact true if (3) is dropped.] *H. Pollard* (New Haven, Conn.).

Dennis, Joseph J., and Wall, H. S. The limit-circle case for a positive definite J-fraction. Duke Math. J. 12, 255–273 (1945). [MF 12597]

This paper continues work by Wall and his students on the theory of convergence of a positive definite "J" continued fraction and discusses the so-called limit-circle case. The authors show that in this case the convergence of the J-fraction or of its reciprocal for one value of z implies the convergence of the J-fraction or of its reciprocal to a function meromorphic for all values of z. The analysis provides new proofs of a number of known theorems on the convergence of continued fractions the elements of which are complex numbers. *W. Leighton* (St. Louis, Mo.).

Wall, H. S. Note on a certain continued fraction. Bull. Amer. Math. Soc. 51, 930–934 (1945). [MF 14463]

The continued fraction

$$\frac{1}{1+}\,\frac{az}{1+}\,\frac{bz}{1+}\,\frac{(a+1)z}{1}\,+\,\frac{(b+1)z}{1}\,+\,\frac{(a+2)z}{1}\,+\cdots$$

with a and b complex, not negative integers or zero, is shown to converge to a meromorphic function in the complex z-plane with the negative real axis and the poles of the limit function deleted. Consequently several continued fraction expansions of integrals are extended to nonreal values of the parameters. *R. P. Boas, Jr.* (Providence, R. I.).

Fourier Series and Generalizations, Integral Transforms

Kober, H. Sur les séries de Fourier. C. R. Acad. Sci. Paris 220, 763–765 (1945). [MF 14066]

A necessary and sufficient condition that a trigonometric series $(*)$ $\sum_{-\infty}^{\infty} a_k e^{ik\theta}$ is a Fourier series is that either (i) there exists a subsequence $\{\sigma_{n_j}\}$ of the $(C, 1)$ means of $(*)$ or (ii) there exists a sequence $\{u(r_j, \theta)\}$ of the Abel means of $(*)$ such that the following holds. For any given positive ϵ there exists a positive δ such that, uniformly with respect to n_j or r_j, $|\int_E \sigma_{n_j}(\theta)d\theta| < \epsilon$ or $|\int_E u(r_j, \theta)d\theta| < \epsilon$ for any set E of measure less than δ. A second necessary and sufficient condition concerning the weak convergence of the sequences $\{\sigma_{n_j}(\theta)\}$ or $\{u(r_j, \theta)\}$ was implicitly stated by Zygmund [Trigonometrical Series, Warsaw-Lwów, 1935, pp. 87–88]. Similar necessary and sufficient conditions are given for a function to be representable as a Fourier transform. [Cf. also Cramér, Trans. Amer. Math. Soc. 46, 191–201 (1939); these Rev. 1, 13.] *P. Civin* (Buffalo, N. Y.).

Kuttner, B. On the Gibbs phenomenon for Riesz means. J. London Math. Soc. 19, 153–161 (1944). [MF 13637]

The results of this paper are related to two previous papers of the author on Riesz means for Fourier series [same J. 18, 148–154 (1943); 19, 77–84 (1944); these Rev. 5, 237; 7, 59]. The main result is that if $\lambda \geqq 2$ the Gibbs phenomenon persists for the means (R, n^λ, k) of the Fourier series of a function with a simple discontinuity, and that if $\lambda > 2$ it persists also for the Abel means (A, n^λ). If $0 < \lambda < 2$ the Gibbs phenomenon vanishes for $k \geqq r(\lambda)$ but not for $k < r(\lambda)$, where $r(\lambda)$ is a function of λ studied in the paper. *R. Salem* (Cambridge, Mass.).

Bosanquet, L. S. The Cesàro summability of the successively derived allied series of a Fourier series. Proc. London Math. Soc. (2) 49, 63–76 (1945). [MF 13709]

In a previous paper [same Proc. (2) 46, 270–289 (1940); these Rev. 1, 329] the author gave necessary and sufficient conditions for the r times differentiated Fourier series of a function $f \epsilon L$ to be, at a given point x, summable $(C, \alpha+r)$, $\alpha \geqq 0$, to sum s. In the present paper a similar problem is solved for the r times differentiated conjugate series. In the simplest case $r=1$ the result may be stated as follows. A necessary and sufficient condition that the series conjugate to the Fourier series of f differentiated term by term should, at the point x, be summable $(C, \alpha+1)$, $\alpha \geqq 0$, to sum s is that, with $g(t) = \{f(x+t)+f(x-t)-2f(x)\}/2t$, (i) the odd function $g(t)$ should be integrable CL over $(0, \pi)$ and its conjugate series summable (C, α) at $t=0$; (ii) $g(t)/t$ should be integrable CL over $(0, \pi)$, and

$$\int_{\to 0(C)}^{\infty} t^{-1} g(t)dt = \tfrac{1}{2}\pi s.$$

(Integrability CL of $g(t)$ over $(0, \pi)$ means that the integral $\int_0^\pi g(t)dt = \lim_{\epsilon \to +0} \int_\epsilon^\pi$ exists by some Cesàro method.) *A. Zygmund* (Philadelphia, Pa.).

Singh, A. N. On divergent Fourier's sine series. Proc. Benares Math. Soc. (N.S.) 5, 41–44 (1943). [MF 14442]

An example of a continuous function whose Fourier sine series is divergent on a set of the second category. *P. Civin* (Buffalo, N. Y.).

Petersen, Richard. On Laguerre polynomials and almost periodic functions. Mat. Tidsskr. B. 1945, 145–150 (1945). (Danish) [MF 14268]

The author considers an almost periodic function whose characteristic exponents form a bounded set. He shows, by means of the Laplace transform, that the function can be developed into a series of Laguerre polynomials. *František Wolf* (Berkeley, Calif.).

Bohr, Harald, and Følner, Erling. On some types of functional spaces. A contribution to the theory of almost periodic functions. Acta Math. 76, 31–155 (1945). [MF 13198]

In this systematic and comprehensive paper, the authors study sets and spaces of generalized a.p. (almost periodic) functions and related functions. Stepanoff, Weyl and Besicovitch norms, $DS_L{}^p[f(x)]$, $DW^p[f(x)]$, $DB^p[f(x)]$ are defined in the usual way, and a general symbol $DG^p[f(x)]$ or $DG[f(x)]$ is used to denote any one of them. A G-a.p. function is defined as the G-limit of a sequence of trigonometric polynomials, while a G-function is simply a function whose G-norm is finite. The G-set (or G-a.p. set) is the set of all G-functions (or G-a.p. functions); if we identify functions of the G-set (or G-a.p. set) whose G-distances from each other are zero, we obtain G-points (or G-a.p. points) which make up the G-spaces (or G-a.p. spaces). Functions at G-distance zero from $f(x)=0$ are called G-zero functions.

The authors investigate completeness or incompleteness of G-spaces and G-a.p. spaces. [There seems to be some overlapping between this part of the paper and the work of Kovanko [cf. Rec. Math. [Mat. Sbornik] N.S. 9(51), 389–401 (1941); C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 117–118 (1941); 43, 49–50, 275–276 (1944); these Rev. 2, 362; 3, 107; 6, 265].] The rest of the main part of the paper is concerned with interrelations among G^p- and G^p-a.p. sets and spaces corresponding to distinct values of p. Here are two general relationships and one characteristic property each for S, W, and B functions. A G^1-a.p. (or G^1-zero) function of finite G^q-norm is G^p-a.p. (or G^p-zero) if $1 \leqq p < q$. Again, if $DG^p[f_p(x)]$ is finite and $DG^1[f_p(x)-f_1(x)]=0$ for $1 \leqq p < P$ $(1 < P \leqq \infty)$, then there is a "through function" $f(x)$ such that $DG^p[f(x)]$ is finite and $DG^1[f_p(x)-f(x)]=0$ for $1 \leqq p < P$. Finally, every S^p- or S^p-a.p. point is an S^1-point; the W^p- and W^p-a.p. spaces are incomplete; and a B^1-a.p. point which contains a B^p-function $(p \geqq 1)$ also contains a B^p-a.p. function. Apart from these and other basic relationships, the authors show by examples (most of which are limit-periodic step-functions) that the various points, sets, and spaces for different values of p are otherwise completely independent. They express this in the characteristic and frequently occurring phrase, "we will show that the following possibilities (which are all those imaginable beforehand) may occur."

An appendix by Følner gives additional examples, in which both types and order differ. *R. H. Cameron.*

Chandrasekharan, K. On Sturm-Liouville series. J. Indian Math. Soc. (N.S.) 8, 109–114 (1944). [MF 13274]

Generalization of some familiar properties of trigonometric Fourier series to the Sturm-Liouville (Fourier) series $\sum a_n \Phi_n(x)$ of a function f in the interval $(0, \pi)$. Here $\{\Phi_n(x)\}$ is the normal orthogonal system of eigenfunctions of the equation

$$\Phi'' + (\rho^2 - l(x))\Phi = 0$$

with boundary conditions $\Phi'(0) - h\Phi(0) = 0$, $\Phi'(\pi) + H\Phi(\pi) = 0$.

To give one example, $a_n\Phi_n'(x)\to\mathfrak{B}(x)/\pi$ (C,α), where $\alpha>1$ and f has a jump $\mathfrak{B}(x)=f(x+0)-f(x-0)$ at x. If, in addition, f is of bounded variation, then $\alpha>0$. This generalizes a result of Fejér [J. Reine Angew. Math. 142, 165–168 (1913)]. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Sinvhal, S. D. On a divergent series of Legendre functions. Proc. Benares Math. Soc. (N.S.) 5, 37–39 (1943). [MF 14441]

Following the lines of Fejér's classical construction for Fourier series, the author constructs a series of Legendre functions, corresponding to a continuous function, which diverges at $x=\pm1$. *H. Pollard* (New Haven, Conn.).

Haller, B. Verteilungsfunktionen und ihre Auszeichnung durch Funktionalgleichungen. Mitt. Verein. Schweiz. Versich.-Math. 45, 97–163 (1945). [MF 13912]

This is a compilation of information, without proofs. The chapter headings are as follows. I. Verteilungsfunktionen. II. Die charakteristischen Funktionen. III. Übersicht über die bekannten Verteilungsfunktionen. IV. Funktionalgleichungen. V. Die Lösungen der Funktionalgleichungen. VI. Das asymptotische Verhalten der Potenzen von Verteilungsfunktionen. Chapter III contains a table of 70 distribution functions, with their Fourier transforms and other information. There is an eight-page bibliography.
R. P. Boas, Jr. (Providence, R. I.).

Widder, D. V. What is the Laplace transform? Amer. Math. Monthly 52, 419–425 (1945). [MF 13755]

This is a well-worded expository note giving the definition of the Laplace transform and showing its relation to power series, Dirichlet series, the Fourier transform, differential equations, the moment problem, the Stieltjes transform and the calculus of differential operators. *H. P. Thielman.*

Saxer, Walter. Über die Laplace-Transformation und ihre Anwendungen. Mitt. Verein. Schweiz. Versich.-Math. 45, 19–29 (1945). [MF 13909]

Expository article, indicating, in particular, applications to the integral equations of renewal theory.

Richmond, D. E. Elementary evaluation of Laplace transforms. Amer. Math. Monthly 52, 481–487 (1945). [MF 14082]

This note gives a systematic outline of elementary methods for constructing tables of Laplace transforms. Several examples are worked out to show the use of the tables derived. The methods used are well within the reach of students familiar with the solution of elementary differential equations. *H. P. Thielman* (Ames, Iowa).

***Blanc, Charles. Sur le calcul des transformées de Laplace de certaines fonctions.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 105–110, Füssli, Zürich, 1945.

The author discusses the evaluation of Laplace integrals $\int_0^\infty e^{-st}F(t)dt$ when $F(t)$ is a step-function, obtaining essentially the usual derivation of Dirichlet series from such integrals. The results are applied to second order differential equations whose solution may have a discontinuous derivative at some points. *H. Pollard.*

Tewari, N. D. A theorem in operational calculus. Proc. Benares Math. Soc. (N.S.) 5, 33–36 (1943). [MF 14440]

It is proved that, if $\varphi_n(p)$ is the Laplace integral of $x^{-n-1}f(x)$, $\Re(n)<0$, then for suitable values of p and m

$$\varphi_{m+n}(p)=\frac{p}{\Gamma(m)}\int_0^\infty \frac{s^{m-1}}{p+s}\varphi_n(p+s)ds.$$

Several applications are given; for example, it is proved that in the case of the zeta functions we have

$$\zeta(k-m,p)=\frac{\Gamma(k)}{\Gamma(m)\Gamma(k-m)}\int_0^\infty s^{m-1}\zeta(k,p+s)ds$$

for $\Re(k)>1$, $\Re(k-m)>1$. *H. Pollard.*

Humbert, Pierre. Une nouvelle correspondance symbolique. C. R. Acad. Sci. Paris 218, 99–100 (1944). [MF 13455]

The author discusses the unilateral Laplace transform of the reciprocal of the gamma function and related functions in terms of functions introduced by Colombo [Bull. Sci. Math. (2) 67, 104–108 (1943); these Rev. 6, 269].
A. E. Heins (Cambridge, Mass.).

Humbert, Pierre, et Poli, Louis. Sur certaines transcendantes liées au calcul symbolique. Bull. Sci. Math. (2) 68, 204–214 (1944). [MF 13418]

The authors derive numerous identities involving the functions discussed by Humbert and Colombo [cf. the preceding review]. *A. E. Heins* (Cambridge, Mass.).

Bayard, Marcel. Correspondance des fonctions de fonctions dans des transformations fonctionnelles définies par généralisation de la transformation de Laplace. C. R. Acad. Sci. Paris 218, 27–29 (1944). [MF 13447]

The author discusses a class of functional transformations which was considered by van der Pol [Physica 1, 521–530 (1934)] and J. P. Schouten [Physica 2, 75–80 (1935)] in connection with the Laplace transform. *A. E. Heins.*

Olevsky, M. Sur une formule sommatoire liée à la transformation de Hankel. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 350–354 (1945). [MF 13754]

Let a_{ij} be integers and let the form $\sum_{i,j=1}^k a_{ij}x_ix_j$ be positive definite. Let $B=\sum_{i,j=1}^k A_{ij}x_ix_j$, where (A_{ij}) is the inverse matrix of (a_{ij}); let D be the determinant of (a_{ij}) and let $r_A(n)$, $r_B(n)$ denote the number of integral solutions of $\sum_{i,j=1}^k a_{ij}x_ix_j=n$, $\sum_{i,j=1}^k A_{ij}x_ix_j=n$, respectively. Let $\varphi(y)$ and

$$\Phi(x)=\int_0^\infty \varphi(y)yJ_p(2\pi xy)dy$$

satisfy suitable continuity and order conditions, where $p=\frac{1}{2}(k-2)$. Then

$$D^{\frac{1}{2}}\{\lim_{x\to0}\varphi(x^{\frac{1}{2}})x^{p/2}+\sum_{n=1}^\infty r_A(n)n^{-p/2}\varphi(n^{\frac{1}{2}})\}$$

$$=\frac{2\pi^{p+1}}{\Gamma(p+1)}\int_0^\infty \varphi(t)t^{p+1}dt+\sum_{n=1}^\infty r_B(n)(n/D)^{-p/2}\Phi((n/D)^{\frac{1}{2}}).$$

The proof uses the k-dimensional Poisson summation formula. Special cases include the one-dimensional Poisson formula $(k=1, a_{11}=1)$ and a formula of Landau [Vorlesungen über Zahlentheorie, vol. 2, Hirzel, Leipzig, 1927, p. 274] involving the number of representations of an integer as the sum of two squares $(p=0, a_{ij}=\delta_{ij})$.
R. P. Boas, Jr. (Providence, R. I.).

Juncosa, Mario L. An integral equation related to Bessel functions. Duke Math. J. 12, 465–471 (1945). [MF 13515]

The integral equation

$$2\int_0^\infty \exp(-\alpha|s-t|-2t)f(t)dt=\lambda f(s), \qquad \alpha>0,$$

arises in the theory of random noise. By an application of standard methods the author shows that the solutions are of the form

$$f(s)=J_\alpha\{2(\alpha/\lambda)^{\frac{1}{2}}e^{-s}\}, \qquad \alpha, \lambda>0,$$

for suitable values of λ. From this he obtains results concerning the completeness of sets of functions $J_\alpha(r_n x)$, where the r_n are roots of other Bessel functions. For example, the set $\{1, J_0(r_n x)\}$ is complete $C(0, 1)$ if the r_n are the positive roots of $J_1(x)$. Many such completeness theorems can be found in the literature, but the use of this integral equation for the purpose is new. *H. Pollard* (New Haven, Conn.).

Krein, M. On a generalization of some investigations of G. Szegö, V. Smirnoff and A. Kolmogoroff. C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 91–94 (1945). [MF 12703]

Suppose that L denotes a linear aggregate of functions $f(e^{i\theta})$ which contains 1 and is such that if f belongs to L then so does $e^{i\theta}f(e^{i\theta})$. The author denotes by L_σ the set of all σ-integrable functions $\phi(\theta)$ such that $\int_0^{2\pi}\phi^2(\theta)d\sigma(\theta)$ is finite and by H_2 the set of all functions $F(e^{i\theta})$ of L^2 which are such that

$$\int_0^{2\pi} F(e^{i\theta})e^{-ni\theta}d\theta=0$$

for $n<0$. He proves that, if σ is such that L is a subset of L_σ, L is not dense in L_σ if and only if (1) $-\infty<\int_0^{2\pi}\log\sigma'(\theta)d\theta$ and (2) there is a function $F(e^{i\theta})$ of H_2 such that

$$\int_0^{2\pi}\{|F(e^{i\theta})|^2/\sigma'(\theta)\}d\theta<\infty$$

and $f(e^{i\theta})F(e^{i\theta})$ belongs to H_2 for any function f of L.

From this result he deduces some other theorems on convergent Blaschke products and on meromorphic functions. The results of this paper are related to some of his earlier results [same C. R. (N.S.) 44, 219–222 (1944); these Rev. 6, 270; cf. also same C. R. (N.S.) 46, 306–309 (1945); these Rev. 7, 61]. *A. C. Offord* (Newcastle-upon-Tyne).

Schwartz, Laurent. Sur certaines familles non fondamentales de fonctions continues. Bull. Soc. Math. France 72, 141–145 (1944). [MF 13224]

Let $u(p)$ be continuous over the n-dimensional space R^n. Then the set of functions (of p) $\{u(\lambda p+q)\}$, where λ varies over all real numbers and q over all points of R^n, fails to span the space of all continuous functions on some compact set $K\subset R^n$ if and only if $u(p)$ is the limit, uniformly on every compact set, of solutions of an equation

$$(1)\qquad \sum_{p_1+\cdots+p_n=m} a(p_1, \cdots, p_n)\frac{\partial^m u}{\partial x_1^{p_1}\cdots\partial x_n^{p_n}}=0$$

with constant coefficients.

If the transformations $\lambda p+q$ are replaced by similitudes, there is a similar theorem with (1) replaced by $\Delta^m u=0$. In this form the result extends to n dimensions a theorem of Choquet and Deny [same Bull. 72, 118–140 (1944); these Rev. 7, 161]. *H. Pollard* (New Haven, Conn.).

Polynomials, Polynomial Approximations

Batschelet, Eduard. Untersuchungen über die absoluten Beträge der Wurzeln algebraischer, insbesondere kubischer Gleichungen. Verh. Naturforsch. Ges. Basel 55, 158–179 (1944). [MF 14668]

The totality of equations $a_0x^n+a_1x^{n-1}+\cdots+a_n=0$ obtained by varying the amplitudes of the a_k, but keeping the moduli $A_k=|a_k|$ fixed, are said to form an equimodular family of equations $F(A_1, \cdots, A_n)$. Let the roots x_k of each equation in F be labelled so that $|x_1|\geqq\cdots\geqq|x_n|$; let $Z_k=\max|x_k|$ and $z_k=\min|x_k|$ with respect to all equations in F, and let $c(n, k)=Z_k/z_k$. Ostrowski [Acta Math. 72, 99–155, 157–257 (1940); these Rev. 1, 323; 2, 342] showed that a constant c_n exists for which $c(n, k)\leqq c_n$ for all k and that $c_2=1+2^{\frac{1}{2}}$, $c_n<0.73\,(n+1)^2$ and $\liminf c_n/n\geqq4/\pi=1.237$. The author shows that $c(n, 1)=c(n, n)=(2^{1/n}-1)^{-1}$ and thus that $\liminf c_n\geqq(\log 2)^{-1}$. The proof is based on the Pellet-Walsh Theorems [Walsh, Ann. of Math. (2) 26, 59–64 (1924)] and an inequality due to Birkhoff [Bull. Amer. Math. Soc. 21, 494–495 (1915)]. *M. Marden* (Milwaukee, Wis.).

Batschelet, Eduard. Über die Schranken für die absoluten Beträge der Wurzeln von Polynomen. Comment. Math. Helv. 17, 128–134 (1945).

Let $x_1, \cdots, x_n$ be the zeros of the polynomial

$$A_0x^n+A_1x^{n-1}+\cdots+A_n$$

arranged so that $|x_1|\geqq\cdots\geqq|x_n|$. On variation only of the arguments of the A_k, the moduli $|x_k|$ of the zeros will vary between limits denoted by ζ_k and $\gamma_n^{(k)}\zeta_k$. Ostrowski showed [Acta Math. 74, 99–155, 157–257 (1940); these Rev. 1, 323; 2, 342] that $\gamma_n^{(k)}\leqq(\frac{3}{2}+\sqrt{2})k(n-k+1)$. In the present paper the author proves that, for fixed k and n, $\max\gamma_n^{(k)}>k(n-k+1)$. This generalizes his earlier result [see the preceding review], which covered only the case that n is odd and $k=(n+1)/2$. The new result is derived after proving, as a lemma, the existence of two real polynomials $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ and $g(x)=b_0x^n+b_1x^{n-1}+\cdots+b_n$ with $|a_p|=|b_p|$ for $p=0, 1, \cdots, n$, such that $f(x)$ has a k-fold zero σ and $g(x)$ an $(n-k+1)$-fold zero τ with $\sigma/\tau>k(n-k+1)$. *M. Marden* (Milwaukee, Wis.).

Ancochea, Germán. Sur l'équivalence de trois propositions de la théorie analytique des polynomes. C. R. Acad. Sci. Paris 220, 579–581 (1945). [MF 14056]

The following three propositions concern a complex polynomial $F(x)$ and the real polynomials $P(x)$ and $Q(x)$ such that $F(x)=P(x)+iQ(x)$ with $P(x)$ and $Q(x)$ having no common factor. (A) All the zeros of $F(x)$ are in one of the half planes $\Re(x)>0$ or $\Re(x)<0$. (B) The zeros of $P(x)$ and $Q(x)$ are real, simple and interlacing. (C) For arbitrary real constants p and q, the zeros of the polynomial $pP(x)+qQ(x)$ are all real. Their equivalence is well-known through the work of Biehler [J. Reine Angew. Math. 87, 350–352 (1879)], Hermite [Bull. Soc. Math. France 7, 128–131 (1880)], Kakeya, and Montel [Mathematica, Cluj 5, 110–129 (1931)]. In the present note the theorems (C)→(A) and (C)→(B) are given new elementary proofs, based on continuity and Descartes' rule of signs. Furthermore, for the more general case that $P(x)$ and $Q(x)$ have as highest common factor $f(x)\neq$constant, the equivalence of the following propositions is stated without proof. (A′) All the zeros of $F(x)$ lie in the half plane $\Re(x)\geqq0$ or $\Re(x)\leqq0$.

(B′) The zeros of $P(x)$ and $Q(x)$ are all real and those of $P(x)/f(x)$ and $Q(x)/f(x)$ are simple and interlacing.
M. Marden (Milwaukee, Wis.).

Ancochea, Germán. Sur les polynomes dont les zéros sont symétriques par rapport à un contour circulaire. C. R. Acad. Sci. Paris **221**, 13–15 (1945). [MF 14234]

If the zeros α_k of a polynomial $f(x)$ are symmetric in the unit circle C, then $|f'(x)/f(x)| \geqq n/2$ for any x on C. This theorem is proved by noting that in

$$f'(x_0)/f(x_0) = \sum \{(x_0-\alpha_k)^{-1} + (x_0 - 1/\bar{\alpha}_k)^{-1}\}$$

for any x_0 on C the terms $(x_0-\alpha_k)^{-1}$ and $(x_0-1/\bar{\alpha}_k)^{-1}$ represent points symmetric in the line $2\Re(x_0 x)=1$. Also, for such polynomials, the derivative with respect to a point t, that is, $F(x, t) = (t-x)f'(x) + nf(x)$, has the properties (1) that, if point t is on C, then t together with the zeros of $F(x, t)$ on C separate the zeros of $f(x)$ which lie on C; and (2) that, if t is not on C, the only zeros of $F(x, t)$ on C are the multiple zeros of $f(x)$. This generalization of Rolle's theorem is stated without proof. *M. Marden* (Milwaukee, Wis.).

Montel, Paul. Remarque sur la note précédente. C. R. Acad. Sci. Paris **221**, 15 (1945). [MF 14235]

By use of a lemma due to Walsh, the first theorem quoted in the preceding review is extended to polynomials $f(x)$ of degree n whose zeros belong to one or both of the two following classes: those interior to C and those forming couples symmetric in C. *M. Marden.*

Gawrilow, L. Über F-Polynome. V. Über K-Fortsetzbarkeit der Polynome. Bull. Soc. Phys.-Math. Kazan (3) **12**, 139–146 (1940). (German. Russian summary) [MF 13904]

Gawrilow, L., und Tschebotaröw, N. Über F-Polynome. VI. K-Polynome mit verschobenem Zentrum. Bull. Soc. Phys.-Math. Kazan (3) **12**, 183–195 (1940). (Russian. German summary) [MF 13908]

Gawrilow has shown that any polynomial $1+a_1z+\cdots+a_nz^n$ can be extended, by adding suitable terms $a_{n+1}z^{n+1}+\cdots+a_mz^m$, so that the roots of the extended polynomial all lie on the unit circle [note IV of this series, same Bull. **8**, 125–129 (1937); for later communications see C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 235–236 (1941); **37**, 246–249 (1942); these Rev. **3**, 236; **6**, 62]. In the first of these papers he gives a simpler proof which provides an explicit determination of the coefficients $a_{n+1}, \cdots, a_m$. If the (unknown) roots of the extended polynomial are $\theta_1, \cdots, \theta_m$, he considers the equations $s_k = \sum_{j=1}^m e^{ik\theta_j}$, where the s_k are expressed in terms of $a_1, \cdots, a_n$ by Newton's formulas. Since no restriction is placed on m, the equations can be solved by a sort of successive approximation argument.

The second paper replaces the unit circle by an arbitrary circle containing the origin in its interior. Here the equations to be solved are more complicated, and use is made of the theory of the trigonometric moment problem.
R. P. Boas, Jr. (Providence, R. I.).

Grünwald, G. On the theory of interpolation. Acta Math. **75**, 219–245 (1943). [MF 13212]

Let $x_1^{(n)}, \cdots, x_n^{(n)}$ be a set of n distinct points in the interval $-1 \leqq x \leqq +1$ and let $\omega_n(x) = (x-x_1^{(n)}) \cdots (x-x_n^{(n)})$ and $v_k^{(n)}(x) = 1-(x-x_k^{(n)})\omega_n''(x_k^{(n)})/\omega_n'(x_k^{(n)})$. The system of points $\{x_k^{(n)}\}$, $k=1, \cdots, n$; $n=1, 2, \cdots$ is said to be normal if $v_k^{(n)}(x) > 0$ for $-1 \leqq x \leqq 1$ and to be strongly normal or ρ-normal if $v_k^{(n)}(x) \geqq \rho > 0$ for $-1 \leqq x \leqq 1$. Fejér introduced and discussed normal point systems. He showed that if $f(x)$ is any continuous function then the Hermite interpolation polynomials, which take the values $f(x_k^{(n)})$ at the points $\{x_k^{(n)}\}$ and whose derivatives assume any uniformly bounded set of values $\{d_k^{(n)}\}$ at the same points, converge uniformly to $f(x)$.

In this paper the author extends some of the results of Fejér by replacing the condition for normal point sets by $v_k^{(n)}(x) \geqq 0$ and by replacing the condition for the $\{d_k^{(n)}\}$ to be uniformly bounded by $|d_k^{(n)}| \leqq n^{\rho-1}$ for ρ-normal systems. He also shows that, if the system of points is ρ-normal and if $f(x)$ satisfies a Lipschitz condition of order greater than $\frac{1}{2}(1-\rho)$, then the Lagrange interpolation polynomials of $f(x)$ converge uniformly in every interval $-1+\delta \leqq x \leqq 1-\delta$, $\delta > 0$. Finally, if

$$l_k(x) = l_k^{(n)}(x) = \omega_n(x)/\{\omega'(x_k^{(n)})(x-x_k^{(n)})\},$$

he shows that for normal point systems $\sum_{k=1}^n l_k^2(x) \to 1$, $n \to \infty$. *A. C. Offord* (Newcastle-upon-Tyne).

Hahn, Wolfgang. Über Orthogonalpolynome mit drei Parametern. Deutsche Math. **5**, 273–278 (1940). [MF 14333]

H. L. Krall has found all orthogonal polynomial sets (including some nonclassical sets) that satisfy a fourth order linear differential equation of form $(*)$ $\sum_{i=1}^4 p_i(x)y^{(i)}(x) = 0$, where $p_i(x)$ is a polynomial of degree not exceeding i, and where only in $p_0(x)$ is the index n of the nth polynomial solution $y_n(x)$ present [Duke Math. J. **4**, 705–718 (1938); Pennsylvania State College Studies, no. 6 (1940); these Rev. **2**, 98]. No comprehensive study has been made of the case where all the coefficients $p_i(x)$ of $(*)$ may involve n, but in the present work four classes of orthogonal polynomials (different from those of Krall) are found satisfying equations of the form $(*)$ with n present in the $p_i(x)$. These classes contain from one to three parameters and are generalizations of the classical polynomials of Jacobi, Laguerre, Hermite and Lommel. Generating functions and second order recursion formulas are found for them.
I. M. Sheffer (State College, Pa.).

Steffensen, J. F. On a class of polynomials. Mat. Tidsskr. B. **1945**, 10–14 (1945). (Danish) [MF 14258]

A number of formulas are given concerning the polynomials $R_\nu^{[\lambda]}(x)$, which may be defined by

$$(t/\varphi(t))^\lambda e^{xt} = \sum_{\nu=0}^\infty t^\nu R_\nu^{[\lambda]}(x),$$

where $\varphi(t)$ is a power series, $\varphi(0)=0$, $\varphi'(0)\neq 0$, and λ is a complex number. These polynomials are connected with the author's "poweroids" [Acta Math. **73**, 333–366 (1941); these Rev. **3**, 236]. They are a special case of Appell sets [P. Appell, Ann. Sci. École Norm. Sup. (2) **9**, 119–144 (1880)]. *R. P. Boas, Jr.* (Providence, R. I.).

Differential Equations

Karapandjitch, M. G. Conditions d'intégralité de l'équation de Riccati. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **26**, 305–313 (1940). [MF 13833]

Applying a contact transformation given by Saltykow to the Riccati equation $(*)$ $y'+Py^2+Qy+R=0$, the author gives the conditions on P, Q and R in order that $(*)$ may

be transformed into one of the forms

$$(x_1y_1'-y_1)^2=y_1+m, \quad (x_1y_1'-y_1)^2=x_1+m,$$

where m is a constant. *F. G. Dressel.*

Brand, Louis. The Lagrange identity as a unifying principle. Amer. Math. Monthly 52, 499–502 (1945). [MF 14085]

The Lagrange identity is $vP(u)-uQ(v)=dR/dx$, where

$$P(u)=P_0(x)u''+P_1(x)u'+P_2(x)u,$$

$Q(v)=(P_0v)''-(P_1v)'+P_2v$ is the adjoint of $P(u)$, and $R=P_0(u'v-uv')+(P_1-P_0')uv$. In this expository paper it is shown how the Lagrange identity leads to the consideration of the Wronskian of two linearly independent solutions of the equation $P(u)=0$, the Liouville differential equation satisfied by this Wronskian, a second solution of $P(u)=0$ when one solution is known, and the general solution of $P(u)=f(x)$. *L. A. MacColl* (New York, N. Y.).

Wuytack, F. Le calcul symbolique des opérateurs linéaires à coefficients variables. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 170–178 (1942). [MF 13654]

It is shown that under suitable conditions an operational calculus can be devised to handle linear differential equations with nonconstant coefficients. *H. Pollard.*

Rosenblatt, A. On the growth of the solutions of ordinary differential equations. Bull. Amer. Math. Soc. 51, 723–727 (1945). [MF 13609]

The author considers the rate of growth of the solutions of the linear differential equation $x''+\psi(t)x'+\phi(t)x=0$. The growth of x is appraised in terms of $\psi(t)$ and $\phi(t)$, generalizing a result of the reviewer for the case where $\psi(t)\equiv 0$ [Duke Math. J. 8, 1–10 (1941); these Rev. 2, 287].

N. Levinson (Cambridge, Mass.).

Minorsky, N. On non-linear phenomenon of self-rolling. Proc. Nat. Acad. Sci. U. S. A. 31, 346–349 (1945). [MF 13933]

In a self-excited system, such as self-rolling of a ship induced by reversing the phase of the control action of an antirolling stabilizing system, the amplitude of the steady state motion cannot be determined by linear theory. Using the equation $\ddot\theta+\omega^2\theta=a\dot\theta+b\dot\theta^2+c\dot\theta^3+d\dot\theta^4$ and assuming the right member to be small, a solution is obtained based on the first perturbation correction. Knowing the experimental as well as the theoretical amplitude of the motion, useful information about Froude's coefficients of resistance can be obtained. *N. Levinson* (Cambridge, Mass.).

Abelé, Jean. Définition cinématique des oscillations de relaxation. J. Phys. Radium (8) 6, 96–103 (1945). [MF 13792]

A generalized form of van der Pol's equation is considered, of the form $z''-\omega\epsilon(1-\phi(z))z'+\omega^2\psi(z)=0$ with ϕ even and ψ odd. The author shows that for a wide range of physically interesting cases a good approximation to the steady state solution is given by

$$z=\cos\theta-f(\sin\theta), \quad \omega t=\int_{\theta_0}^{\theta}\frac{-\sin\theta}{dz/d\theta}\,d\theta,$$

where in each case f is a suitably chosen function. The method also may be used for the generalized Rayleigh equation. *N. Levinson* (Cambridge, Mass.).

Abelé, Jean. Définition cinématique des oscillations de relaxation. C. R. Acad. Sci. Paris 220, 511–515 (1945). [MF 14050]

An abstract of the article reviewed above.

N. Levinson (Cambridge, Mass.).

Rocard, Yves. Attaque des systèmes vibrants par des moyens non linéaires. Rev. Sci. (Rev. Rose Illus.) 80, 359–363 (1942). [MF 13991]

The approximate solution of a certain system with two degrees of freedom is obtained by use of the method of variation of parameters, the parameters being slowly changing functions. *N. Levinson* (Cambridge, Mass.).

Parodi, Hippolyte, et Parodi, Maurice. Les équations de relaxation, cas particulier des équations de la marche d'un train. Revue Sci. (Rev. Sci. Illus.) 81, 110–120 (1943). [MF 13807]

The derivation of an equation of the type associated with relaxation oscillations is given and graphical solutions are discussed. *N. Levinson* (Cambridge, Mass.).

Maratschkow [Maračkov, V.]. Über einen Liapounoffschen Satz. Bull. Soc. Phys.-Math. Kazan (3) 12, 171–174 (1940). (Russian. German summary) [MF 13906]

The system $\dot x_j=f_j(x_1,\cdots,x_n,t)$, $j=1,\cdots,n$, has as a solution $x_j=0$, $j=1,\cdots,n$, for $t=t_0$. The f_j are uniformly bounded for $t\geqq t_0$ and $\sum x_j^2\leqq c$, $c>0$. If there is a definite function $V(x_1,\cdots,x_n,t)$ such that its derivative $\dot V$, expressed as a function of x and t by using the differential equations, is also definite and of opposite sign to V, then the solution $x_j=0$, $j=1,\cdots,n$, is stable. This generalizes a result of Liapounoff which required V to have an arbitrarily small upper bound. *N. Levinson* (Cambridge, Mass.).

Rytov, S. M. An extension of the limits of applicability of the small parameter method. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 181–184 (1945). [MF 13740]

The author considers applications of the perturbation method which are of physical interest but of no mathematical novelty. *N. Levinson* (Cambridge, Mass.).

Lee, H. C. On the factorization method for quantum mechanical eigenvalue problems. Chinese J. Phys. 5, 89–104 (1944). [MF 13762]

The factorisation method of solving the Sturm-Liouville eigenvalue problem was introduced by Schrödinger [Proc. Roy. Irish Acad. Sect. A. 46, 9–16 (1940); 183–206 (1941); these Rev. 1, 277; 3, 245] and developed by Infeld [Phys. Rev. (2) 59, 737–747 (1941); these Rev. 2, 364]. No general theory underlying the method has so far been given. In the present paper, the author formulates explicit conditions (too lengthy to give here) under which the method applies and gives formulas enabling him to write out at once the solution of any eigenvalue problem satisfying these conditions. *E. T. Copson* (Dundee).

Boulanger, J. Sur l'équation différentielle linéaire et homogène du second ordre. Bull. Soc. Roy. Sci. Liége 9, 89–110 (1940). [MF 13039]

The author considers the equation $\{K(x)y'\}'+L(x)y=0$ on an interval (a, b) in which the functions $K'(x)$ and $L(x)$ are continuous and $0<m\leqq K(x)\leqq M$, $0<n\leqq L(x)\leqq N$, with m, M, n and N constants. The particular matter at issue is the existence of a solution that is positive within the interval and such that $y(a)=0$, $y'(b)=0$. Conditions for

this are deduced by methods of comparison and successive approximation. It is, for instance, found to be necessary that $(m/N)^{\frac{1}{2}}<b-a<\frac{1}{2}\pi(M/N)^{\frac{1}{2}}$. *R. E. Langer.*

Boulanger, J. Remarques sur certaines équations différentielles du second ordre. Bull. Soc. Roy. Sci. Liége 9, 213–223 (1940). [MF 13047]

The paper considers the equation $\{K(x)z'\}'+\eta(x)=0$, or its transformation by a change of dependent variable, $y''+A(x)y=0$, with $A(x)$ positive and continuous for $a\leqq x\leqq b$. A number of special cases are given in which the conditions $y(a)=0$, $y'(b)=0$ imply $y(x)\equiv 0$.

R. E. Langer (Madison, Wis.).

Boulanger, J. Sur une équation différentielle linéaire du second ordre. II. Bull. Soc. Roy. Sci. Liége 10, 32–56 (1941). [MF 13050]

[Part I appeared in the same Bull. 8, 538–560 (1939).] The paper is concerned with the equation $y''+A(x)y=0$, in which $A(x)$ is continuous and positive. In particular, it deals with the existence of an integral $Y_1(x)$ which is not identically zero and fulfills the conditions

$$\alpha Y_1(a)+\beta Y_1'(a)=0,\quad \gamma Y_1(b)+\delta Y_1'(b)=0,$$

relative to an interval $a\leqq x\leqq b$, the coefficients α, β, γ, δ being constants. It is observed that $Y_1(x)\equiv 0$ if a solution $Y(x)$ exists fulfilling the conditions

$$\alpha Y(a)+\beta Y'(a)=m\geqq 0,\quad \gamma Y(b)+\delta Y'(b)=n\geqq 0,$$

$m^2+n^2\neq 0$, and this existence is investigated by the method of successive approximations. *R. E. Langer.*

Boulanger, J. Sur l'équation différentielle linéaire du second ordre. Bull. Soc. Roy. Sci. Liége 12, 372–389 (1943). [MF 13148]

In earlier papers the author has considered the conditions under which a differential equation

$$A(x)y''+B(x)y'+C(x)y=0,$$

with $A(x)>0$, $C(x)>0$ on (a, b), admits an integral (not identically zero) which vanishes at $x=a$ and $x=b$. In the present paper this is extended to the case in which $A(x)>0$, but $C(x)$ changes sign on (a, b). The method depends essentially upon the use of elementary transformations of the dependent variable under which the equation is transformed into one of the earlier type. *R. E. Langer.*

Boulanger, J. Sur l'équation différentielle du troisième ordre. Bull. Soc. Roy. Sci. Liége 9, 110–116 (1940). [MF 13189]

The paper is concerned with the uniqueness of the solution of a differential equation

$$y'''=A(x)y''+B(x)y'+C(x)y,$$

fulfilling boundary conditions

$$y(a)=\alpha,\quad y'(a)=\beta,\quad y(b)=\gamma,\qquad a<b.$$

This is assured if the conditions $y_1(a)=y_1'(a)=y_1(b)=0$ imply $y_1\equiv 0$. The relation

$$\int_a^b\{\lambda(x)y_1y_1''\}'dx=0,$$

with an unspecified $\lambda(x)$, is transformed into the form

$$P\{y_1'(b)\}^2+\int_a^b\{Q(x)y_1^2+R(x)y_1'^2\}dx=0.$$

If $\lambda(x)$ can be chosen to make P, $Q(x)$ and $R(x)$ positive the conclusion is at hand. An application to the equation $y'''+\eta(x)y=0$ with $0\leqq\eta(x)\leqq M$ shows $y(x)$ to be unique if $b-a\leqq(\frac{3}{2}M)^{\frac{1}{2}}$. *R. E. Langer* (Madison, Wis.).

Boulanger, J. Sur l'équation différentielle du troisième ordre. Bull. Soc. Roy. Sci. Liége 10, 223–233 (1941). [MF 13062]

The paper gives some conditions under which an equation $y'''=A(x)y''+B(x)y'+C(x)y$ admits no more than one solution fulfilling the boundary conditions $y(a)=m$, $y'(a)=p$, $y(b)=n$, where $a<b$ and the coefficients are differentiable on (a, b). By changes of variable and integration by parts the equation is made to assert the vanishing of a sum of terms. The conditions come from the fact that the terms cannot all be of the same sign. Samples of conditions obtained are $A\geqq 0$, $B'-AB-2C<0$ and

$$9A''-18AB'-27B'+4A^3+18AB+54C\geqq 0.$$

R. E. Langer (Madison, Wis.).

Boulanger, J. Sur l'équation différentielle linéaire et homogène du quatrième ordre. Bull. Soc. Roy. Sci. Liége 11, 220–233 (1942). [MF 13100]

The paper deals with the differential equation

$$y^{iv}=A(x)y'''+B(x)y''+C(x)y'+D(x)y.$$

By changes of variable and integrations by parts, a number of sets of complicated inequalities upon differential expressions in the A, B, C, D are deduced, under which the conditions $y(a)=y'(a)=0$, $y(b)=y'(b)=0$ imply that $y(x)\equiv 0$ on (a, b). The inequalities must hold over the entire interval.

R. E. Langer (Madison, Wis.).

Strutt, M. J. O. Bounds for the characteristic values of Hill problems. I. Characteristic values with smallest moduli. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 83–90 (1943). (Dutch. German, English and French summaries) [MF 13896]

This is the first of a series of papers on boundary value problems of the following type: the characteristic functions w satisfy the equation

$$L(w)+\{\lambda+\gamma\phi(z)\}w(z)=0, \tag{1}$$

where (2) $L(w)=d^2w/dz^2+p(z)w(z)$; p and ϕ are real periodic functions with period ζ and have convergent Fourier series; λ and γ are real parameters; and the boundary conditions are (3) $w(z_1)=\sigma w(z_0)$, $w'(z_1)=\sigma w'(z_0)$, $z_1=z_0+\zeta$, in which σ is a given constant, either real or complex with modulus unity. Such a problem is called a Hill problem; because of (3) Hill problems are not in general self-adjoint. Since λ and γ are two independent parameters, one more relation is needed to make (1), (3) a characteristic value problem. The author considers three cases: (i) the ratio of λ and γ is given; (ii) λ is given; (iii) γ is given. The characteristic values determine (i) the actual values of λ and γ, (ii) γ, (iii) λ.

In this paper the problem is rewritten in the form of an integral equation

$$w(x)=h\int_{z_0}^{z_1}\{\lambda_0+\gamma_0\phi(y)\}G(x, y)w(y)dy, \tag{4}$$

where $G(x, y)$ is the Green's function of the problem considered, $\gamma=h\gamma_0$, $\lambda=h\lambda_0$, $\lambda_0=0$ for (ii) and $\gamma_0=0$ for (iii). For all three problems G is expressed in terms of solutions f_1, f_2 of (1) satisfying the initial conditions $f_1(z_0)=f_2'(z_0)=1$,

$f_1'(z_0)=f_2(z_0)=0$. By the usual methods, lower bounds for the first (lowest) characteristic value are derived in terms of

$$I=\int_{z_0}^{z_1}\int_{z_0}^{z_1} G(x, y)dxdy.$$

Explicit expressions for G are given for $p(z)=0$. Numerical results include graphs of $(\zeta/2\pi)^2\sqrt{I}$ plotted against $(\zeta/2\pi)^2\lambda$ for $\sigma=\pm1, \pm i, \pm5, \pm100$. These graphs can be used to obtain bounds for $|\gamma|$ as a function of λ. *A. Erdélyi.*

Strutt, M. J. O. Bounds for the characteristic values of Hill problems. II. Characteristic values of arbitrary order. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 97–104 (1943). (Dutch. German, English and French summaries) [MF 13897]

The well-known maximum-minimum properties of characteristic values of self-adjoint boundary value problems are summarised. Attention is then turned to boundary value problems which are not self-adjoint. Let w satisfy the differential equation $L(w)+\lambda g(z)w(z)=0$ and (3) of the preceding review; w^* satisfies the adjoint equation $L^*(w^*)+\lambda g(z)w^*(z)=0$ and the boundary conditions $w^*(z_1)=\sigma^{-1}w^*(z_0)$, $w^{*\prime}(z_1)=\sigma^{-1}w^{*\prime}(z_0)$. The maximum-minimum properties of the characteristic values are formulated in terms of the expressions

$$M=M[w]=-\int_{z_0}^{z_1} w(z)L^*(w^*)dz, \quad N=\int_{z_0}^{z_1} w(z)w^*(z)g(z)dz.$$

In particular, if $|\sigma|=1$ then w and w^* are conjugate complex. In this case the maximum-minimum property of the nth positive or negative characteristic value is proved. The case where $g(z)$ changes sign in (z_0, z_1) is considered. A table shows the sign of variation of characteristic values with varying M and N, provided that at least one of these forms is definite (that is, has the same sign for all admissible functions w). As an application to Hill problems, bounds for positive and negative characteristic values of any order are given and illustrated in a diagram. *A. Erdélyi.*

Strutt, M. J. O. Characteristic curves of Hill problems. I. General shape of the curves. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 153–162 (1943). (Dutch. German, English and French summaries) [MF 13898]

Given σ, Hill's problem [cf. the two preceding reviews] establishes a relation between λ and γ which can be interpreted as a characteristic curve in the (λ, γ)-plane. For such curves, λ and γ are analytic functions of one another. From the results of the second part of this paper [cf. the following review] it is concluded that to every value of σ there is a denumerable sequence of such curves. Double points occur only when $\sigma=\pm1$ and singular points of higher multiplicity can never occur. The author derives expressions for $d\lambda/d\gamma$ and $d^2\lambda/d\gamma^2$ along the characteristic curves, the actual formulae being those well-known from perturbation theories. In particular, if (λ, γ) is a double point, an alternative formula gives the two values of $d\lambda/d\gamma$ as the roots of a secular equation of degree two. For $|\sigma|=1$, λ is an entire function of γ and if, in addition, $\phi(z)$ does not change sign, γ is also an entire function of λ. In this case λ and γ are monotonic functions of one another. For $|\sigma|\neq1$ similar statements hold in restricted parts of the (λ, γ) plane. The mutual position of the characteristic curves is elucidated by considering their intersections with a straight line which makes an angle α with the positive λ-axis. A curve for which $\sigma=1$ is called a periodic curve, one for which $\sigma=-1$ a semi-periodic curve. If $1+\phi(z)\tan\alpha$ does not change sign with variable z, then, when λ is made to increase from $-\infty$, a periodic curve is met first, then two semi-periodic ones, then two periodic ones, and so on. Between any two curves of different kinds lies one and only one curve for every value of σ such that $|\sigma|=1$. The author also discusses the shape of the characteristic curves in the vicinity of the λ-axis and gives descriptive properties of the curves in the whole (λ, γ)-plane, including information about direction of tangents and sign of curvature. *A. Erdélyi.*

Strutt, M. J. O. Characteristic curves of Hill problems. II. The asymptotic form of the curves. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 212–222 (1943). (Dutch. German, English and French summaries) [MF 13899]

The asymptotic solutions of (1) [cf. the third preceding review] are discussed for large values of $(|\lambda|+|\gamma|)\zeta^2$. If $p(z)+\lambda+\gamma\phi(z)$ has zeros, Stokes's phenomenon will occur and complicate the picture. The author assumes that there are two simple zeros in every period and uses R. E. Langer's method to obtain asymptotic solutions in this case. From these he derives asymptotic formulae for the characteristic values of Hill's problem. From the asymptotic formulae it is seen that two periodic curves are followed by two semi-periodic ones and vice versa; two consecutive unlike curves (one periodic and one semi-periodic) are asymptotically very near each other and all the curves corresponding to complex σ with $|\sigma|=1$ cluster between them. With constant γ and increasing λ the clustering becomes denser. Asymptotic expressions are also obtained for $d\lambda/d\gamma$ and $d^2\lambda/d\gamma^2$ along the curves. The investigation is extended to the case where $p+\lambda+\gamma\phi$ has any even number of simple zeros in a period. *A. Erdélyi* (Edinburgh).

Strutt, M. J. O. Characteristic functions of Hill problems. I. Completeness of the sets of periodic and almost periodic characteristic functions. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 488–496 (1943). (Dutch. German, English and French summaries) [MF 13900]

[For notations cf. the four preceding reviews.] In this and the following part $|\sigma|=1$ is assumed, so that w and w^* are conjugate complex. The characteristic value belonging to the normalised characteristic function w_k is λ_k; the relation

$$\int_{z_0}^{z_1} w_m(z)w_n^*(z)g(z)dz=0$$

holds for $m\neq n$. Straight lines in the (λ, γ)-plane along which $g=1+\phi\tan\alpha>0$ for all z are called lines of the first kind, those along which g changes sign lines of the second kind. Along lines of the first kind there is a denumerable infinity of characteristic values, positive or negative according to the sign of M; along lines of the second kind N is indefinite and there is a denumerable infinity of both positive and negative characteristic values. For lines of the second kind the author puts

$$f_n(z)=f(z)-\sum_{-n}^{n}{}'\pm(\lambda_k/|\lambda_k|)w_k(z)\int_{z_0}^{z_1} f(x)w_k^*(x)g(x)dx,$$

where the prime indicates that the term corresponding to $k=0$ is to be omitted; the upper or the lower sign is to be used according as M is positive or negative definite. The

completeness relation

$$\int_{s_0}^{s_1} f_n(s)f_n^*(s)g(s)ds \to 0, \qquad n\to\infty,$$

for functions $f(s)$ which are twice differentiable and satisfy the boundary conditions, is deduced from the inequality

$$\int_{s_0}^{s_1} f_n f_n^* g ds \leqq M[f]/\lambda_{n+1}$$

for positive definite M and a similar inequality for negative definite M. The proof remains valid for lines of the first kind. As an application, Green's function and the iterated nuclei are used to obtain estimates for the lowest characteristic value by the well-known root-squaring method.

A. Erdélyi (Edinburgh).

Strutt, M. J. O. Characteristic functions of Hill problems. II. Expansion formulas in series of periodic and of almost periodic characteristic functions. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 584–591 (1943). (Dutch. English, German and French summaries) [MF 13901]

[Cf. the preceding review.] Asymptotic expressions for the characteristic functions and characteristic values are derived for lines of the first kind, and for lines of the second kind with the additional assumption that g has only two simple zeros in every period. In particular, it is shown in these cases, and stated also in the case of any even number of zeros of g in a period, that the characteristic functions are uniformly bounded. The expansion in terms of characteristic functions is proved first for a function $f(s)$ which is twice differentiable and satisfies the boundary conditions. From the asymptotic representation of the characteristic functions by means of trigonometric functions the author concludes that the conditions on $f(s)$ can be relaxed in the same way as with Fourier series. As an application, the bilinear expansions of the Green's function and its iterates are given. There is also the well-known expansion in terms of characteristic functions of the solution of the inhomogeneous equation $L(w)+\lambda g(s)w(s)=f(s)g(s)$.

A. Erdélyi (Edinburgh).

Arley, Niels, and Borchsenius, Vibeke. On the theory of infinite systems of differential equations and their application to the theory of stochastic processes and the perturbation theory of quantum mechanics. Acta Math. 76, 261–322 (1945). [MF 13203]

The systems studied are of the form $\mathbf{Y}'(t)=\mathbf{A}(t)\mathbf{Y}(t)+\mathbf{B}(t)$, where $\mathbf{A}$ is an infinite matrix, $\mathbf{B}$ a given, $\mathbf{Y}$ an unknown column-vector. The elements of $\mathbf{A}$ and $\mathbf{B}$ are supposed continuous in $T_0<t<T_1$. Existence and uniqueness are proved under the following three conditions: (1) $\mathbf{A}$ is "absolutely exponentiable" in (T_0, T_1), by which is meant that, putting $K=\max|A(\tau)|$ for $T_0\leqq\tau\leqq t$, one has

$$\exp\{\mathbf{K}(t-T_0)\}=\sum \mathbf{K}^\nu(t-T_0)^\nu/\nu!<\infty;$$

(2) only those solutions $\mathbf{Y}(t)$ are considered for which $\exp\{\mathbf{K}(t-T_0)\}\cdot\mathbf{G}<\infty$, $\mathbf{G}=\max \mathbf{Y}(\tau)$ for $T_0\leqq\tau\leqq t$; (3) $\mathbf{B}$ is subject to a condition analogous to (2). Several conditions on $\mathbf{A}$ are stated which are sufficient to ensure (1). For example, (1) follows from the uniform boundedness of either the row or column sums of $\mathbf{A}$.

For stochastic processes [Kolmogoroff, Math. Ann. 104, 415–458 (1931)] only the case $\mathbf{B}=0$, $\mathbf{A}=(\mathbf{\Pi}-\mathbf{I})\mathbf{p}$ is of interest, where $\mathbf{p}=\{p_i\}$ is a nonnegative column-vector, $\mathbf{I}$ the identity, and $\mathbf{\Pi}$ a matrix (of conditional transition probabilities) with $\pi_{ij}\geqq 0$, $\pi_{ii}=0$, $\sum_j\pi_{ij}=1$. However, the theory now requires that the elements Y_i of $\mathbf{Y}$ satisfy (*) $Y_i\geqq 0$ and (‡) $\sum_i Y_i=1$. An existence proof for the case of uniformly bounded p_i has been given by Feller [Math. Ann. 113, 113–160 (1936)]. In applications unbounded p_i occur [Lundberg, On Random Processes and their Application to Sickness and Accident Statistics, Stockholm Thesis, 1940; these Rev. 2, 230]. Then (*) but not necessarily (‡) holds. The author finds a sufficient condition for (‡). [Another general existence proof and necessary and sufficient conditions for (‡) were given by Feller [Trans. Amer. Math. Soc. 48, 488–515 (1940); these Rev. 2, 101]; his method applies to more general processes, leading to Stieltjes integral equations of which differential equations are a special case.] Many examples are given to illustrate the various "pathologies" (like multiplicity of solutions) which become possible if some of the conditions are not fulfilled.

The applications to quantum mechanics are primarily of physical interest.

W. Feller (Ithaca, N. Y.).

Snapper, Ernst. Partial differentiation and elementary divisors. Ann. of Math. (2) 46, 388–408 (1945). [MF 13427]

A system S of linear, homogeneous, partial differential equations, with constant coefficients, in a single unknown, gives rise to an auxiliary polynomial ideal $\mathfrak{a}$ when the operators $\partial/\partial t_1, \cdots, \partial/\partial t_n$ which appear in S are replaced by variables $x_1, \cdots, x_n$. Each associated prime ideal $\mathfrak{p}$ of $\mathfrak{a}$ furnishes a solution exp ($\mathfrak{p}$) of S. The "differental exponent δ of $\mathfrak{p}$ with respect to $\mathfrak{a}$" is defined as unity plus the degree of a polynomial f of highest degree such that the product f exp ($\mathfrak{p}$) is a solution of S. The relation of δ to other (algebraic) exponents connected with $\mathfrak{a}$ is investigated. The number of initial conditions which can be formally imposed on a solution derived from f exp ($\mathfrak{p}$) is computed.

The paper generalizes familiar results of the theory of one ordinary differential equation. The coefficient domain for most of the discussion is the field of complex numbers.

H. Levi (New York, N. Y.).

Choquet, Gustave, et Deny, Jacques. Sur quelques propriétés de moyenne caractéristiques des fonctions harmoniques et polyharmoniques. Bull. Soc. Math. France 72, 118–140 (1944). [MF 13223]

For a preharmonic function $f(x, y)$, defined on a square network D, and for an arbitrary square S formed by elementary squares of D, the average of $f(x, y)$ on the diagonals of S, vertices of S excluded, and the average of $f(x, y)$ on the sides of S, vertices of S excluded, are equal. This observation leads to the question, answered affirmatively in the present paper, as to whether the Gauss characterization of harmonic functions, in terms of mean-values on circles, can be extended to other sets. Thus it is shown that a function $f(x, y)$, continuous in a finite simply-connected domain D, is harmonic there if and only if, for each regular polygon P of p sides in D, the mean-value of $f(x, y)$ on the perimeter of P is equal to the mean-value of $f(x, y)$ on the radii of P.

The method used depends on the determination of distributions λ of mass on a bounded set E_0 such that the Stieltjes integral (1) $\int_E f(x, y)d\lambda$ vanishes for all functions $f(x, y)$ harmonic in the plane and for all sets E similar to E_0 and similarly charged; these distributions are the ones for which the corresponding potentials vanish identically in the neigh-

borhood of infinity. The further investigation of the vanishing of the integral (1), for functions $f(x, y)$ which are not necessarily harmonic, leads to characterizations of polyharmonic functions. *E. F. Beckenbach.*

Gassmann, Fritz. Vom Gravitationsfeld des inhomogenen Rotationsellipsoides im Aussenraum. Vierteljschr. Naturforsch. Ges. Zürich 90, 36–43 (1945). [MF 13965]

For a piecewise continuous and bounded distribution of mass in the ellipsoid of revolution E,

$$\frac{x^2+y^2}{b^2+D^2}+\frac{z^2}{b^2}-1=0,$$

where D is real or pure imaginary, the author discusses the gravitational potential in terms of a corresponding rotational ellipsoidal coordinate system [cf. E. Heine, Handbuch der Kugelfunktionen, vol. 2, Reimer, Berlin, 1881]. In particular, the convergence of a series expansion giving a solution of the second, or Neumann, boundary value problem for the exterior of E is established. *E. F. Beckenbach.*

Lagrange, René. Sur une classe d'harmoniques associés aux cyclides de révolution. Bull. Soc. Math. France 72, 169–177 (1944). [MF 13226]

Continuing his previous researches [Acta Math. 71, 283–315 (1939); these Rev. 1, 238], the author seeks to obtain harmonic functions associated with cycloids of revolution, the meridian curves of which are quadrics with one real focus and two further real or conjugate imaginary foci on the axis of revolution. From the parametric equations of the surfaces the associated harmonics are obtained, the individual products containing two functions satisfying Lamé's differential equations of potential type. These equations are written in their algebraic form and solved by polynomials. Their properties and those of the characteristic parameter values are derived from the theory of these equations. *M. J. O. Strutt* (Eindhoven).

Hadamard, J. Remarques sur le cas parabolique des équations aux dérivées partielles. Publ. Inst. Mat. Univ. Nac. Litoral 5, 3–11 (1945). [MF 13719]

Methods developed for Laplace's equation are applied to three problems for the heat equation (*) $\partial^2u/\partial x^2=\partial u/\partial y$. (1) An example is given to show that the Cauchy problem for (*) does not depend continuously on the given initial conditions. (2) In the Dirichlet type problem for (*) the solution is unique if the given initial values of u on the boundary are continuous; the author shows that, if the initial values are continuous, except perhaps for one point, and if u is bounded, the solution is unique. (3) Finally, the author points out that the prolongment problem for (*) can be handled in the same manner as E. E. Levi [Rend. Circ. Mat. Palermo 24, 275–317 (1907)] handled the same problem for Laplace's equation. *F. G. Dressel.*

Higgins, Thomas James. Temperature distribution in toroidal electrical coils of rectangular cross section. J. Franklin Inst. 240, 97–112 (1945). [MF 12870]

A formula is derived for the steady temperatures $T(x, y)$ in a long prism of rectangular cross section $|x|\leqq a$, $|y|\leqq b$ when the heat equation has the form $k_1T_{xx}+k_2T_{yy}=-c_1-c_2T$, where the k's and c's are constants. Thus the rate at which heat is generated within the prism is a linear function of the temperature. The transfer of heat at the four faces is assumed to take place according to Newton's law. At the face $x=a$, for example, the boundary condition is $T(a, y)-T_1=-K_1T_x(a, y)$, where the constant T_1 is the temperature of the air in contact with that face and K_1 is a constant. Fourier analysis is used to obtain a series that represents $T(x, y)$. The author demonstrates the use of the formula by computing temperatures in an electric coil with a rectangular cross section. The corresponding problem is treated when the coil has the form of a hollow cylinder of finite length and is also illustrated for the case of an actual coil. *R. V. Churchill* (Ann Arbor, Mich.).

Craggs, J. W. Heat conduction in semi-infinite cylinders. Philos. Mag. (7) 36, 220–222 (1945). [MF 13347]

A formula for the temperatures $v(r, \theta, z, t)$ in a wedge $0<\theta<\theta_0$, $r>0$, $z>0$, is derived when $v=1$ on the plane $\theta=\theta_0$, $v=0$ on the other two planes and the initial temperature of the wedge is zero. The temperature formula is also written for this solid when $v=1$ initially and $v=0$ on all boundary surfaces. Finally, under the latter conditions a formula is written when the solid is a cylindrical wedge $0<r<a$, $0<\theta<\theta_0$, $z>0$. The method consists of the successive use of these processes: a Laplace transformation with respect to t, a Fourier sine transformation with respect to t, a Fourier sine transformation with respect to z and a modified form of separation of variables with respect to r and θ. *R. V. Churchill* (Ann Arbor, Mich.).

Wagner, Carl. Wärmeleitungsprobleme für Systeme mit beheizten Rohren und Hohlkugeln in einer unendlich ausgedehnten Umgebung. Ing.-Arch. 14, 398–409 (1944). [MF 13376]

The infinite solid outside an infinitely long circular cylinder is initially at temperature zero throughout. The surface of the cylinder is then kept at a constant temperature. The author develops simple approximate formulas for the flux of heat across the cylindrical surface as a function of time. He also develops corresponding formulas in the case of the infinite solid outside a sphere. Sources of heat at the surfaces are considered too; however, the treatment of one of these cases rests upon the author's formula (5.1), which appears to be incorrect. *R. V. Churchill.*

Barrer, R. M. Diffusion in spherical shells, and a new method of measuring the thermal diffusivity constant. Philos. Mag. (7) 35, 802–811 (1944). [MF 12954]

A heat conduction problem for spherical shells is treated which the author immediately reduces to the classical problem of the heat conduction in a bar where the initial temperature is given while the ends of the bar are kept at a constant given temperature. The main value of the paper seems to lie in the applications. *E. H. Rothe.*

Jaeger, J. C. Diffusion in turbulent flow between parallel planes. Quart. Appl. Math. 3, 210–217 (1945). [MF 13533]

Problems of evaporation of moisture from the earth's surface into turbulent air were treated by W. G. L. Sutton [Proc. Roy. Soc. London. Ser. A. 182, 48–75 (1943); these Rev. 5, 69]. The equation of diffusion, under appropriate assumptions, is

$$(1)\qquad \partial^2\chi/\partial z^2+\{(1-2p)/z\}\partial\chi/\partial z=\partial\chi/\partial x,$$

where χ is the concentration of moisture and x is measured in the direction of mean flow parallel to the earth's surface $z=0$. Sutton considered the semi-infinite region $z>0$ with various boundary conditions. The author derives solutions

of (1) by means of the Laplace transformation for regions $z>0$ and $0<z<1$, under a variety of boundary conditions. He includes some discussion of his results. The problem of evaporation is only one of several physical interpretations of these problems. The function χ may represent, for example, the steady temperatures in the turbulent fluid.

R. V. Churchill (Ann Arbor, Mich.).

Sneddon, Ian N. The Fourier transform solution of an elastic wave equation. Proc. Cambridge Philos. Soc. 41, 239–243 (1945). [MF 13422]

This is a sequel to the author's previous paper [same Proc. 44, 27–43 (1945); these Rev. 6, 229]. The equation for the deflection w of a thin uniform plate of thickness $2h$ but infinite extent is

$$b^2(\partial^2/\partial x^2+\partial^2/\partial y^2)^2w+\partial^2w/\partial t^2=8bp(x, y)\psi'(t),$$

where $b^2=D/2\rho h$ and x, y are the rectangular coordinates in the flow of the plate. By the Fourier transform

$$W(\xi, \eta; t)=(1/2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}w(x, y; t)e^{i(\xi x+\eta y)}dxdy$$

the differential equation becomes

$$d^2W/dt^2+\lambda^2W=8bP(\xi, \eta)\psi'(t),$$

where $P(\xi, \eta)$ is the Fourier transform of $p(x, y)$ and $\lambda^2=b^2(\xi^2+\eta^2)^2$. The initial conditions will completely determine $W(\xi, \eta; t)$. Then the Fourier inversion formula gives $w(x, y; t)$ from W. As in the previous paper, the advantage of this method over the usual Laplace transform of the time variable is the directness in obtaining the result for initial displacement or applied external forces.

H. S. Tsien (Pasadena, Calif.).

Petrashen, G. I. The oscillations of an isotropic elastic sphere. C. R. (Doklady) Acad. Sci. URSS. (N.S.) 47, 172–176 (1945). [MF 13739]

The author expands the required vector solution in terms of products of scalar functions of the radius r and an orthonormal set of vector functions of (θ, φ) which he has discussed in a previous paper [same C. R. (N.S.) 46, 266–269 (1945); these Rev. 7, 16], and thereby obtains linear differential equations for the scalar function. These are easily solved, so that he obtains a complete solution for both forced and free oscillations. [It should be noted that these solutions are the familiar ones which occur in the theory of the electromagnetic oscillations within a spherical cavity.]

H. Feshbach (Cambridge, Mass.).

Difference Equations, Special Functional Equations

Wintner, Aurel. The linear difference equation of first order for angular variables. Duke Math. J. 12, 445–449 (1945). [MF 13512]

The author discusses solutions of $f(x+\omega)-f(x)=F(x)$, where ω is irrational, $F(x)$ is of period 1 and of class L_1, and we seek solutions $f(x)$ of the same class satisfying the equation almost everywhere. He gives necessary conditions and also sufficient conditions for the existence of such solutions. Moreover, he discusses the behavior of solutions, as ω varies, for a fixed $F(x)$ which is analytic along the axis and is not a trigonometric polynomial. He finds that, if such an $F(x)$ satisfies the necessary condition that its average vanishes, then for almost all ω (including all algebraic irrational numbers), the (unique up to a constant) solution is also analytic along the axis. On the other hand, there is always an uncountable set of transcendental values of ω for which there is no L_1 solution of period 1 at all, and intermediate cases also arise. The paper is based on Liouville's theory of transcendental numbers.

R. H. Cameron.

***Valeiras, Antonio. Two new types of linear functional equations.** Memorias sobre Matematicas (1942–44) por Antonio Valeiras, pp. 81–100. Buenos Aires, 1944. (Spanish) [MF 12387]

The equations are of the form

$$a_0(x)F(x)+a_1(\Omega x)F(\Omega x)+\cdots+a_n(\Omega^n x)F(\Omega^n x)=k(x).$$

Here Ω is a given (cyclic) operator acting on the variable x and satisfying $\Omega^{n+1}=1$. The $a_i(x)$ and $k(x)$ are given; $F(x)$ is unknown. The author shows how to obtain the general solution (from the system of linear equations obtained by repeated substitution of Ωx for x in the equation). The cases where the rank of the matrix $\|a_i(\Omega^k x)\|$ is, respectively, $n+1$, n, 1 are carried out explicitly. Extensions to more general functional equations (involving, for example, noncyclic groups) are indicated. [Some of the extensions have been covered by similar methods by M. Ghermanescu, Bull. Soc. Math. France 68, 109–128 (1940); Mathematica, Timişoara 18, 37–54 (1942); these Rev. 3, 297; 4, 145.]

F. John (New York, N. Y.).

Ghermanescu, Michel. Sur une classe d'équations fonctionnelles du premier ordre. Acta Math. 75, 191–218 (1943). [MF 13211]

The author calls an equation for a function $F(z)$ of the form (1) $F(\theta)=\varphi(z, F(z))$, where φ is a given function and $z'=\theta(z)$ a given substitution, "a functional equation of 1st order." An "iterative constant" is a function $u(z)$ invariant under the substitution θ. The paper is concerned with the determination of the functional equations for which there exists a unique relation of the form (2) $\phi(F_1, \cdots, F_n)=u(z)$ yielding an iterative constant u for any solutions F_k of (1). First order differential equations may be considered as limiting cases of (1), and it is known that the only such equations with a relation (2) [where u=constant] are the linear and Riccati equations.

The author shows that for equations (1) n cannot exceed the value 4. For $n=4$ the functional equation can be reduced by a substitution $U=U(F)$ to the form

$$F(\theta)=(AF+B)/(CF+D),$$

where A, B, C, D are functions of z. In that case ϕ becomes the cross ratio of the four solutions. For $n=3$ (1) is similarly reducible to a linear equation $F=AF+B$.

F. John (New York, N. Y.).

Montel, Paul. Sur deux systèmes d'équations fonctionnelles. Mathematica, Timişoara 21, 10–11 (1945). [MF 13970]

Extract from a letter of the author indicating that the solution of the system

$$\varphi(x+y)=\varphi(x)\varphi(y)-\psi(x)\psi(y),$$
$$\psi(x+y)=\psi(x)\varphi(y)+\varphi(x)\psi(y)$$

treated by Anghelutza [Bull. Sci. École Polytéch. Timişoara 11 (1943); these Rev. 5, 72] follows also from results of the author [Ann. Sci. École Norm. Sup. (3) 48, 65–94 (1931)].

F. John (New York, N. Y.).

GEOMETRY

Anning, Norman H., and Erdös, Paul. Integral distances. Bull. Amer. Math. Soc. 51, 598–600 (1945). [MF 12821]

The authors show that for any n there exist noncollinear points $P_1, \cdots, P_n$ in the plane such that all distances P_iP_j are integers; but there does not exist an infinite set of noncollinear points with this property. [Cf. the following review.] *I. Kaplansky* (Chicago, Ill.).

Erdös, Paul. Integral distances. Bull. Amer. Math. Soc. 51, 996 (1945). [MF 14475]

The paper reads as follows.

"In a note under the same title [see the preceding review] it was shown that there does not exist in the plane an infinite set of noncollinear points with all mutual distances integral.

"It is possible to give a shorter proof of the following generalization: if A, B, C are three points not in line and $k=[\max(AB, BC)]$, then there are at most $4(k+1)^2$ points P such that $PA-PB$ and $PB-PC$ are integral. For $|PA-PB|$ is at most AB and therefore assumes one of the values $0, 1, \cdots, k$, that is, P lies on one of $k+1$ hyperbolas. Similarly P lies on one of the $k+1$ hyperbolas determined by B and C. These (distinct) hyperbolas intersect in at most $4(k+1)^2$ points. An analogous theorem clearly holds for higher dimensions." *I. Kaplansky* (Chicago, Ill.).

Goodman, A. W., and Goodman, R. E. A circle covering theorem. Amer. Math. Monthly 52, 494–498 (1945). [MF 14084]

The authors prove the following theorem. Let there be given n circles in the plane, with radii $r_1, \cdots, r_n$, such that no line separates the circles. Then they can be covered by a circle of radius at most $\sum_{i=1}^n r_i$. The proof is simple and elegant. Some analogous problems are also discussed.

P. Erdös (Stanford University, Calif.).

Rutishauser, H. Über Punktverteilungen auf der Kugelfläche. Comment. Math. Helv. 17, 327–331 (1945).

The author investigates the problem of finding n points on the surface of the unit sphere so that the smallest (spherical) distance between any two of them is maximal. Denote this distance by Δ_n. Fejes proved [Jber. Deutsch. Math. Verein. 53, 66–68 (1943)] that

$$\cos\Delta_n \geqq \tfrac{1}{2}\{\cot^2 \tfrac{1}{6}n\pi/(n-2)-1\}.$$

This formula is exact for $n=3, 4, 6, 12$, in which cases the points are vertices of the regular polyhedra. Fejes also proved that for $n=8$ and 20 the regular polyhedra do not give a solution of the problem. In fact, he proved that $0.2101<\cos\Delta_8<0.3333$, $0.6558<\cos\Delta_{20}<0.7454$; 0.3333 and 0.7454 are the values given by the cube and the dodecahedron. The author finds polyhedra which improve the upper bounds to 0.2612 and 0.6861. The question of whether these bounds are best possible is undecided. *P. Erdös.*

van Veen, S. C. Constructions on a solid sphere. The vertices of the five regular inscribed polyhedra. Mathematica, Zutphen. A. 13, 15–20 (1944). (Dutch) [MF 14315]

Unkelbach, Helmut. Die kantensymmetrischen, gleichkantigen Polyeder. Deutsche Math. 5, 306–316 (1 plate) (1940). [MF 14336]

Consider a finite polyhedron whose edges are of equal length and lie in planes of symmetry, while its faces are convex polygons which do not penetrate one another. It need not be convex, but will always be Eulerian. The author finds just twenty such figures: the five Platonic solids, a triangular dipyramid, a pentagonal dipyramid, a triakistetrahedron (consisting of four regular tetrahedra stuck onto respective faces of a fifth), a triakisoctahedron (Kepler's "stella octangula"), a triakisicosahedron, a tetrakishexahedron (half an octahedron stuck onto each face of a cube), two pentakisdodecahedra (the second having its pyramids scooped out from the faces of the dodecahedron, as in fig. 26, Taf. VIII of M. Brückner, "Vielecke und Vielfläche," Teubner, Leipzig, 1900), two hexakisoctahedra, two hexakisicosahedra, the familiar rhombic dodecahedron and triacontahedron, and a remarkable rhombic hexecontahedron. The last is illustrated by a photograph of a model. Its dihedral angles, at the two types of edge, are $2\pi/5$ and $6\pi/5$. Its faces have the same shape as those of the triacontahedron, of which it is actually a stellation. Similarly, the second pentakisdodecahedron is a stellated icosahedron [Coxeter, Du Val, Flather and Petrie, The Fifty-nine Icosahedra, Univ. Toronto Studies, Math. Ser. no. 6, 1938, plate IX: "$\mathrm{Ef}_1\mathrm{g}_1$"]. *H. S. M. Coxeter* (Toronto, Ont.).

Merz, K. Mehrfache Kreuzhaube. Vierteljschr. Naturforsch. Ges. Zürich 87, 193–198 (1942). [MF 13966]

Alternate sides of a horizontal $2n$-gon $ABCDE\cdots A_1B_1C_1\cdots$ are joined to two points S and O on the vertical line through its center by $2n$ triangles $ABS, BCO, CDS, DEO, \cdots$ and the remaining sides of these triangles are joined together by n quadrangles $ASA_1O, BSB_1O, \cdots$ having a common diagonal SO which is a singular line (winding line), not counted as an edge. For simplicity, O is taken at the center of the $2n$-gon. When the top face of a $2n$-gonal prism is replaced by such a cross-cap, the resulting polyhedron has $4n+2$ vertices, $10n$ edges, $5n+1$ faces and hence characteristic $3-n$. The case when $n=2$ was described in an earlier paper [same Vierteljschr. 85, 51–57 (1940); these Rev. 2, 260]. *H. S. M. Coxeter* (Toronto, Ont.).

Walker, A. G. A model of a hyperboloid of one sheet and its asymptotic cone. Edinburgh Math. Notes no. 35, 20–23 (1945). [MF 14448]

***Birkhoff, George D. One-dimensional metric geometry and the equation $f(x+y)=f(x)+f(y)$.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 169–183, Füssli, Zürich, 1945.

The author gives a set of six simple postulates for the "purely algebraic" part of Euclidean one-dimensional geometry, which is obtained when order relations are ignored and only a finite number of points are considered at the same time. A mild postulate of order is then added [P 7: The set of points may be so arranged in linear order that, for any P, Q, R, S ($Q\neq P$, $S\neq R$), if $PQ=RS$ then either $P<Q$, $R<S$ or $Q<P$, $S<R$]. It is shown by examples that this is still not the ordinary Euclidean line. [It is unfortunate that the author had no opportunity for proof-reading; for example, the conclusion of P 1* should surely read "then either $x_P<x_Q$, $x_R<x_S$ or $\cdots$."] *H. S. M. Coxeter.*

Cartan, Élie. La notion d'orientation dans les différentes géométries. Bull. Soc. Math. France 69, 47–70 (1941). [MF 13239]

The reference here is to geometrical orientation as opposed to topological orientation, and the distinction is emphasised

throughout. The method of approach is to consider in a geometry, as defined by its group G, the group of stability g of a given subspace. A set of elements of g is said to be connected if any two elements of the set can be linked by a series of transformations within the set, such that each transformation differs infinitesimally from the one following it. Thus g may split into a number of connected sets $g_0, g_1, g_2, \cdots$, where g_0 is that set containing the identity transformation; that is, g may be written as a sum of cosets with regard to g_0. For example, in the group of stability of a line D in elementary Euclidean geometry, translations parallel to D, rotations about D and combinations of these generate g_0; if a rotation about a line perpendicular to and intersecting D is denoted by r, then the remaining elements of g may be written in the form $rg_0=g_1$. For the given subspace to be geometrically orientable in G it is necessary and sufficient that $g_0\neq g$.

Numerous illustrative examples are chosen from real and complex Euclidean and projective geometry. In particular, if the invariant figure is a ruled quadric surface, the group of stability $g=g_0+g_1+g_2+g_3$, where, if λ, μ are the parameters associated with the two systems of generators, g_0 is typified by the transformation $\lambda'=\lambda$, $\mu'=\mu$; g_1 by $\lambda'=-\lambda$, $\mu'=-\mu$; g_2 by $\lambda'=\lambda$, $\mu'=-\mu$; and g_3 by $\lambda'=-\lambda$, $\mu'=\mu$.

G. de B. Robinson (Toronto, Ont.).

Nehring, O. Über ein Dreiecksproblem. J. Reine Angew. Math. 186, 70–77 (1944). [MF 13406]

The problem is to find the locus of the point whose pedal triangle, for a given triangle, is isosceles. The author obtains the well-known result that the locus consists of three circles, but he fails to identify these circles as the Apollonian circles of the basic triangle and spends about two-thirds of his paper in proving classical properties of those circles, the Brocard diameter, the symmedian point, etc., without identifying any of these geometric elements. The remainder of the paper considers properties of the centroid of the pedal triangle. For example, if a point M describes a straight line, then the centroid G of the pedal triangle of M, for a given basic triangle, also describes a straight line and the two ranges $(M\cdots)$, $(G\cdots)$ are projective. [The property that the feet of three lines equally inclined to the respective sides of a triangle and passing through the same point of the circumcircle lie on a straight line is attributed by the author to Boymann (1849) instead of to Poncelet [Traité des Propriétés Projectives des Figures, Paris, 1822, p. 270]. He refers to the Simson line as the "Steinersche Gerade."]

N. A. Court (Norman, Okla.).

Mandan, Ram. A symmetrical figure of circles and points [Grace configuration]. Proc. Lahore Philos. Soc. 7, 52 (1945). [MF 14277]

The author establishes the existence of a configuration of 2^m circles and 2^m points with $m+1$ circles through each point and $m+1$ points on each circle. [There are some confusing misprints for powers and binomial coefficients.]

I. Kaplansky (Chicago, Ill.).

Bottema, O. The problem of the five points. Mathematica, Zutphen. A. 13, 1–4 (1944). (Dutch) [MF 14317]

This is the problem of locating a point P from which five given points in the projective plane will subtend given cross ratios. Calling the five points A, B, C, D, E, take ABC as triangle of reference and DE as the unit line $[1, 1, 1]$. Then the coordinates of P are found to be inversely proportional to the differences $P(BDCE)-A(BDCE)$, $P(CDAE)-B(CDAE)$, $P(ADBE)-C(ADBE)$. When D and E are the absolute points of the Euclidean plane, this becomes Snellius's problem: to locate a point P from which the sides of a given triangle ABC will subtend given angles α, β, γ. The present result thus includes Tienstra's solution of that problem: the barycentric coordinates of P are inversely proportional to $\cot\alpha-\cot A$, $\cot\beta-\cot B$, $\cot\gamma-\cot C$.

H. S. M. Coxeter (Toronto, Ont.).

Bottema, O. The figure of four planes in R_5. I. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 53, 30–37 (1944). (Dutch. German, English and French summaries) [MF 14307]

Four lines a, b, c, d in projective 3-space have, as absolute invariants, the ratios $U:V:W$, where $U=(ab)(cd)$, $V=(ac)(db)$, $W=(ad)(bc)$, and $(ab)=\sum p_{ij}^{(a)}p_{kl}^{(b)}$, summed over the even permutations $ijkl$ of 1234. In general, the four lines have two transversals, in accordance with Segre's symbol [11]. But if $U^{\frac{1}{2}}+V^{\frac{1}{2}}+W^{\frac{1}{2}}=0$, they either have only one transversal (repeated) or belong to a regulus, and the appropriate symbol is [2] or [(11)]. These remarks are exhibited as the case $m=1$ of a general theory of four subspaces R_m in projective R_{2m+1}. When $m=2$ we have four planes in a 5-space, and the author finds six possible arrangements; [111] with three distinct transversals, [21] with one double transversal and one single, [(11)1] with a regulus of transversals and an isolated one, [3] with a triple transversal, [(21)] with a regulus of transversals one of which is double, and [(111)] with ∞^2 transversals.

H. S. M. Coxeter (Toronto, Ont.).

Bottema, O. The figure of four planes in R_5. II. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 53, 53–57 (1944). (Dutch. German, English and French summaries) [MF 14308]

It was observed by Voss [Math. Ann. 13, 168–174 (1878)] that $U^{\frac{1}{2}}+V^{\frac{1}{2}}+W^{\frac{1}{2}}=0$ (in the notation of the preceding review) is the condition for four lines a, b, c, d to touch a twisted cubic curve; and it was observed by Grace [Proc. Cambridge Philos. Soc. 25, 421–432 (1929)] that $U^{1/n}+V^{1/n}+W^{1/n}=0$ is the condition for four planes a, b, c, d in 5-space to osculate a rational normal quintic. The author gives a simple proof of the latter result, and shows further that this is a case of type [111], that is, that four osculating planes of the quintic have three distinct (and in fact real) transversals. [The analogous result for tangents of a cubic in 3-space was obtained by Sturm, Arch. Math. Phys. (3) 23, 1–10 (1914).]

He goes on to prove that two pairs of polar planes with respect to a quadric in 5-space are of type [111], [(11)1] or [(111)]. It follows that, in elliptic 5-space, two planes have either three distinct common perpendiculars, or a regulus and an isolated one, or a set of ∞^2. In the two latter cases it is natural to say that the two planes are "half Clifford parallel" or "Clifford parallel," respectively.

H. S. M Coxeter (Toronto, Ont.).

Bottema, O. The geometry of a certain group of projective transformations. Nieuw Arch. Wiskde. 21, 73–80 (1941). (Dutch) [MF 14295]

This paper describes a new metric for affine plane geometry, the "distance" between two points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ being defined as $(A_1A_2)=-(A_2A_1)=x_1y_2-y_1x_2$ (which is proportional to the area of the triangle OA_1A_2,

where O is the origin). It follows that, for three collinear points, $(A_2A_3)+(A_3A_1)+(A_1A_2)=0$. Dually, the "angle" between two lines $l_1[p_1, q_1]$ and $l_2[p_2, q_2]$ (not through O) is $(l_1l_2)=-(l_2l_1)=p_1q_2-q_1p_2$, so that, for three concurrent lines, $(l_2l_3)+(l_3l_1)+(l_1l_2)=0$. The sides and angles of a triangle ABC now satisfy the "trigonometrical" formulas $a/A=b/B=c/C$, $a+b+c=bcA$, etc. Turning to differential geometry, the author finds that, if A_1 and A_2 are two neighboring points on a curve, where the tangents are l_1 and l_2, then the "curvature" at A_1 is $\lim_{A_2\to A_1}(l_1l_2)/(A_1A_2)$. The curves of constant curvature are the conics with center O; in fact, the curvature of $\alpha x^2+2\beta xy+\gamma y^2=1$ is $\alpha\gamma-\beta^2$. The "radius" may be defined as the reciprocal of the curvature. In this sense, the circumradius of triangle ABC is found to be $4a^2b^2c^2/(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.
H. S. M. Coxeter (Toronto, Ont.).

Benneton, Gaston. Sur les configurations harmoniques. C. R. Acad. Sci. Paris 220, 548–550 (1945). [MF 14053]

A configuration (in complex projective 3-space) is said to be harmonic if its points are the poles of its planes with respect to a quadric. Taking the quadric in the form $x^2+y^2+z^2+t^2=0$, we may represent the point (x, y, z, t) and its polar plane $[x, y, z, t]$ by the quaternion $A=x+yi+zj+tk$. Then the condition for a point (or plane) A and a plane (or point) B to be incident is $S(A\bar{B})=0$. It is pointed out that two general quaternions P and Q yield eight associated points $P, Pi, Pj, Pk, Q, iQ, jQ, kQ$, and that Möbius' configuration 8_4 (of two tetrahedra, mutually inscribed and circumscribed) can be represented as $1, i, j, k, V, iV, jV, kV$, where V is a pure quaternion such as $i+j+k$. Again, the eight quaternions $1\pm i\pm j\pm k$ represent a pair of desmic tetrahedra, which is a harmonic 8_3. Adding the third tetrahedron $1, i, j, k$, we obtain Stephanos' 12_6, which is not harmonic; but when this is combined with the second 12_6, $1\pm i, 1\pm j, 1\pm k, j\pm k, k\pm i, i\pm j$, we obtain a harmonic 24_9.
H. S. M. Coxeter (Toronto, Ont.).

Benneton, Gaston. Sur les configurations de Kummer et de Klein. C. R. Acad. Sci. Paris 220, 640–642 (1945). [MF 14059]

The author uses the notation of his earlier paper [Bull. Sci. Math. (2) 68, 190–192 (1944); these Rev. 7, 69] to express some classical results on configurations. [See Hudson, Kummer's Quartic Surface, Cambridge University Press, 1905, pp. 49–50, 79–80.] He remarks that the collineation group of Klein's 60_{15} configuration, of order $16\cdot 6!$, is generated by sixty harmonic homologies whose centers and axial planes are the vertices and opposite faces of the fifteen "fundamental tetrahedra." *H. S. M. Coxeter.*

Momet, Pierre. Sur les transformations anallagmatiques. Revue Sci. (Rev. Rose Illus.) 80, 200–208 (1942). [MF 13813]

The author scrutinizes the proposition, due to Goursat and rediscovered by others, that any anallagmatic transformation is the product of four inversions of which the first two are orthogonal to the last two [Ann. Sci. École Norm. Sup. (3) 6, 9–102 (1889)]. He restates it in the following corrected form: "Toute transformation anallagmatique est le produit d'une rotation elliptique et d'une rotation orthogonale, à l'exception d'un seul type, qui est le produit d'une rotation parabolique et d'une inversion orthogonale." This result is used to classify anallagmatic transformations and to study the various ways in which such transformations may be decomposed into inversions and anallagmatic rotations. *N. A. Court* (Norman, Okla.).

Apéry, Roger. Recherches sur quelques propriétés anallagmatiques. Rev. Sci. (Rev. Rose Illus.) 80, 347–358 (1942). [MF 13990]

de Saussure, René. Sur la représentation réelle d'une sphère imaginaire au moyen de l'espace réglé. C. R. Séances Soc. Phys. Hist. Nat. Genève 60, 36–39 (1943). [MF 14214]

Revised summary of a paper in Amer. J. Math. 18, 304–346 (1896). "Le résumé a surtout pour but de préciser et fixer la nomenclature."

Rossier, Paul. Démonstration projective de l'équation des foyers conjugués. C. R. Séances Soc. Phys. Hist. Nat. Genève 59, 207–208 (1942). [MF 14208]

Humbert, Pierre. Bitétraèdres de l'espace attaché à l'opérateur Δ_3. Bull. Sci. Math. (2) 68, 50–59 (1944). [MF 13256]

The author studies the geometry in space where the distance d from the origin is defined by the equation $d^3=x^3+y^3+z^3-3xyz$. Consider a pyramid with vertices S, O, A, C. Take a point B on the segment AC and draw lines SB and OB. The resulting geometric configuration is called a bitetrahedron. The center of gravity of five equal masses placed at the points S, O, A, B, C is the intersection of forty-nine lines with remarkable properties. Defining a triplet of lines as mutually orthogonal if the sum of the products of the corresponding direction numbers is zero, it is shown that there exist bitetrahedra whose opposite triplets of edges are all mutually orthogonal. Finally, the author considers a sphere (defined by a cubic equation of special type) which contains sixty-five interesting points of the bitetrahedron. *J. DeCicco* (Chicago, Ill.).

Martin, D. Some properties of the curve of constant bearing. Edinburgh Math. Notes no. 35, 4–9 (1945). [MF 14445]

Kowalewski, Gerhard. Räumliche Mercatorkoordinaten. J. Reine Angew. Math. 186, 65–69 (1944). [MF 13403]

Mercator coordinates as defined by the author are ordinary geographical coordinates in space with latitude β replaced by $\beta^*=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\beta)$. The Laplace spherical harmonics are expressible in simple fashion in terms of these coordinates. *S. B. Myers* (Ann Arbor, Mich.).

Fabricius-Bjerre, Fr. On L. Eckhart's axonometric method. Mat. Tidsskr. B. 1944, 1–16 (1944). (Danish) [MF 13594]

Treating the problem analytically, the author develops a method of deriving axonometric maps of an object if two projections are given. The method contains Eckhart's [Akad. Wiss. Wien, S.-B. IIa. 146, 51–56 (1937); Z. Verein. Deutsch. Ingenieure 1938, 447–448] and Arvesen's [Norske Vidensk. Selsk. Forh. 11, 72–74, 78–81 (1938)] methods as special cases. *W. Feller* (Ithaca, N. Y.).

Graf, Ulrich. Anaglyphen aus parallelprojizierten Teilbildern. Deutsche Math. 5, 317–321 (1 plate) (1940). [MF 14337]

Two parallel projections of a spatial object may be used to form an anaglyph if all the lines joining corresponding

points are parallel. Corresponding points are the two images of the same point in space. Projections on two different planes, as well as two parallel projections on the same plane, are equally suitable. When such a drawing is viewed through an anaglyphoscope a relief perspective of the object is seen.
E. Lukacs (Cincinnati, Ohio).

Bachmann, W. K. Méthode de la connexion des images et théorie des erreurs de l'orientation relative. Schweiz. Z. Vermessgswes. Kulturtech. 43, 98–108, 125–131, 160–168, 173–175, 206–211, 231–236 (1945). [MF 14000]

Bachmann, W. K. Influence de la courbure de la terre sur les triangulations aériennes. Schweiz. Z. Vermessgswes. Kulturtech. 43, 9–15 (1945). [MF 13998]

Convex Domains, Integral Geometry

Alexandroff, A. Existence of a convex polyhedron and of a convex surface with a given metric. Rec. Math. [Mat. Sbornik] N.S. 11(53), 15–65 (1942). (Russian. English summary) [MF 12823]

Let K be a topologic complex homeomorphic to a 2-sphere; a topologic equivalent m of K made up of Euclidean triangles is called a metrization of K, and m is said to be convex if the sum of the angles surrounding each vertex of K is at most 2π. The main theorem is that every convex metric is uniquely realizable by a convex polyhedron. The proof, by induction on the number of vertices of K, proceeds by defining the space M_0 of all metrizations, not necessarily convex, of K; since any metrization is determined by giving the lengths of all edges of K, and these need only satisfy the condition that the sum of two sides of a triangle exceeds the third, M_0 can be considered as the inside of a solid angle in Euclidean k-space, k being the number of edges of K. The set M of convex metrizations is a submanifold of M_0. It is shown that each component of M contains polyhedrally realizable metrizations; from this it follows readily that the set of all polyhedrally realizable convex metrics coincides with the set M of all convex metrics. A second theorem is proved to the effect that any metric of nonnegative curvature defined upon a 2-sphere can be realized by a convex closed surface; the statement is due to H. Weyl and a proof to H. Lewy, but the present proof is new. *H. Wallman* (Cambridge, Mass.).

Alexandroff, A. D. The inner metric of a convex surface in a space of constant curvature. C. R. (Doklady) Acad. Sci. URSS (N.S.) 45, 3–6 (1944). [MF 12569]

Proofs are not given, but are stated to be similar to those in Euclidean space [same C. R. (N.S.) 30, 103–106 (1941); these Rev. 2, 302; and the paper reviewed above]. A convex surface is the boundary of a convex body. A metric ρ of a space R is called inner if, for each pair of points $X,Y\epsilon R$, $\rho(XY)$ is the greatest lower bound of the lengths of curves connecting X and Y. If R is a 2-dimensional surface in a 3-space $\mathfrak{R}$ of constant curvature K, R is convex relative to the metric of $\mathfrak{R}$ under conditions which are somewhat long in statement. We quote the following theorem. "A metric on R defined by a line-element with twice differentiable coefficients is convex relative to the metric of $\mathfrak{R}$ if and only if the Gaussian curvature of the given line element is nowhere less than K." The definition of tangent cone to a 2-space in the 3-space is obvious. The following results are stated. In order that a metric 2-space R be isometric to a convex surface in the 3-space $\mathfrak{R}$ of constant curvature K, the following conditions are necessary: (1) the metric of R must be inner; (2) at each point R must possess a tangent cone; (3) R must be convex relative to the metric of $\mathfrak{R}$. If R satisfies the same three conditions, then each of its points has a neighborhood isometric to a convex surface in $\mathfrak{R}$. If, in addition, R is homeomorphic to a sphere, then R is isometric to a closed convex surface in $\mathfrak{R}$. *A. B. Brown* (Flushing, N. Y.).

Alexandroff, A. D. Complete convex surfaces in Lobachevskian space. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 113–120 (1945). (Russian. English summary) [MF 12768]

All theorems refer to a given Lobachevskian space of curvature K_0. By a complete convex surface is meant a connected component of the boundary of a convex solid; also included are twice-covered plane convex closed regions. Theorem: a convex surface is homeomorphic to a region on a sphere, and for any region on a sphere there exists a complete convex surface homeomorphic to it. Theorem: if a complex of plane polygons is homeomorphic to a sphere and the sum of the angles about each of its vertices is at most 2π, then there exists a convex polyhedron capable of a subdivision isomorphic and isometric to this complex. The proofs are stated to be not essentially different from those for the case of Euclidean 3-space [C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 103–106 (1944); these Rev. 2, 302]. Theorem: let a line element with curvature at least K_0 be given in a region G on a sphere. The metric defined in G by the line element being complete, it is realizable by means of a complete convex surface. The following theorem uses the terminology of the paper reviewed above. In a region G on a sphere, let an inner complete metric be given which has a tangent cone at every point and is convex with respect to the metric of curvature K_0. Then there exists a complete convex surface realizing this metric. *A. B. Brown.*

Alexandroff, A. D. Isoperimetric inequalities for curved surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 235–238 (1945). [MF 14021]

Let D be a domain homeomorphic to a closed circular disk on a surface with line element $Edu^2+2Fdudv+Gdv^2$, where E, F, G have continuous first derivatives, $EG-F^2>0$ and the boundary of D has finite length. The curvature in D is said to be at most k if for every geodesic triangle in D the ratio of the spherical excess and the area is at most k. If the curvature in D is at most $k\leqq 0$ then the area of D does not exceed the area of a circle D_0 with the same perimeter as D in the hyperbolic or Euclidean plane of curvature k. If the curvature in D is at most $k>0$, the perimeter of D is at most $2\pi k^{-\frac{1}{2}}$ and the shortest connection in D is unique, then the area of D does not exceed the area of a circle D_0 with the same perimeter as D on a hemisphere of curvature k. In either case equality holds only when D and D_0 are isometric. *H. Busemann* (Northampton, Mass.).

Alexandroff, A. D. Curves on convex surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 315–317 (1945). [MF 14015]

As angle between two geodesics G_1, G_2 issuing from a point p on a convex surface S in E^3 define the angle ϕ between their tangents at p (which exist), measured on the

tangent cone of S at p. If x_i is the geodesic distance of the point q_i on G_i from p and z the geodesic distance of q_1 and q_2, then $\cos\phi=\lim_{x_i\to 0}(x_1^2+x_2^2-z^2)/2x_1x_2$. Let C be a closed Jordan curve on S bounding an open domain D; let P be a simple closed geodesic polygon in D; let $\tau(P)$ be the sum of the angles of P measured on the other side of P from C. If P converges to C, then $\tau(P)$ converges to a number $\tau_D(C)$. If $W(D)$ denotes the spherical image of D, then the Gauss-Bonnet theorem $W(D)=2\pi-\tau_D(C)$ holds. Simply connected regions D on S which contain with any two points a shortest connection are characterized by the fact that the (analogously defined) curvatures $\tau_D(C_1)$ of the subarcs C_1 of the boundary of D are all negative. No proofs are given.

H. Busemann (Northampton, Mass.).

✶Schwerdtfeger, H. The Isoperimetric Problem. University of Adelaide, South Australia, 1945. ii+14 pp.

Lecture to the Adelaide University Mathematical Society.

Angeletti, Yves. Plans et droites d'appui des corps convexes. Bull. Soc. Roy. Sci. Liége 9, 117–119 (1940). [MF 13040]

Let C be the surface of a convex body in 3-space. If each plane of support of C has a single point in common with C, then the unit sphere S can be mapped continuously on C by means of parallel planes of support [cf. Bonnesen and Fenchel, Theorie der Konvexen Körper, Springer, Berlin, 1934, p. 15]. A "convex domain on S" contains with any two points the great circle arcs of length less than or equal to π connecting them [cf. Santaló, Revista Union Mat. Argentina 8, 113–125 (1942); Duke Math. J. 9, 707–722 (1942); Math. Notae 4, 11–40 (1944); these Rev. 4, 169, 252; 6, 15]. A "convex domain on C" is the image of one on S. If such a domain is not equal to C, it is part of the image of a hemisphere, and through each point on its boundary goes the image of a suitable great circle as a curve of support.

Notation: M is a point set on C; $\bar{M}$ is the smallest convex domain on C containing M; M^* is the boundary of $\bar{M}$ omitting all the images of open great circle arcs on that boundary; $E(M)$ ($\mathfrak{E}(M)$) is the intersection of all the half-spaces that are bounded by the planes of support of C in the points of M and (do not) contain C. Thus $C\subset E(M)$; $E(M)=E(M')\leftrightarrow M=M'$; $\mathfrak{E}(M)=\mathfrak{E}(M')\leftrightarrow\bar{M}=\bar{M}'$; $\mathfrak{E}(M)\neq 0\leftrightarrow C-\bar{M}\neq 0$; if $\mathfrak{E}(M)\neq 0$, then $\mathfrak{E}(M)=\mathfrak{E}(\bar{M})=\mathfrak{E}(M^*)$. A second convex surface C' and a set $M'\subset C'$ exist such that $E(M)=\mathfrak{E}(M')$ and $\mathfrak{E}(M)=E(M')$ if and only if $M=M^*$. The common interior planes of support of two disjoint convex surfaces C and C' envelop two curves $M=M^*\subset C$ and $M'=M'^*\subset C'$ which satisfy the above relations. If a segment is bounded by a point on C and one on C', and if the straight line through it is a line of support of both C and C', then the segment is called a segment of support. The set of these segments is equal to $e-(E(M)+E(M'))$, where e is the smallest convex body that contains C and C'. Finally, the common boundary of $E(M)$ and $e-E(M)$ is described.

P. Scherk (Saskatoon, Sask.).

Blanc, Eugène. Sur les domaines plans appuyables. C. R. Acad. Sci. Paris 220, 238–240 (1945). [MF 13498]

Let E_1, E_2 be plane domains which possess locally, but uniformly, twice differentiable supporting functions $H_i(\phi)$ of period 2π. Denote by E_i' the domain symmetric to E_i with respect to the origin and put $E_3=\frac{1}{2}(E_1+E_2')$, $E_4=\frac{1}{2}(E_1-E_2')$. Here the sum and difference are defined by vector addition, and E_3, E_4 have local supporting functions $H_3(\phi)$, $H_4(\phi)$ with $H_1(\phi)=H_3(\phi)+H_4(\phi)$, $H_2(\phi+\pi)=H_3(\phi)-H_4(\phi)$. Let S_i be the algebraic area of E_i, S_{12}' the mixed area of E_1 and E_2', S_{34} the mixed area of E_3 and E_4. Then the relations for the H_i yield $S_1+S_2=2(S_3+S_4)$, $S_1-S_2=4S_3$, $S_{12}'=S_3-S_4$, $S_{12}'^2-S_1S_2=4(S_{34}^2-S_3\cdot S_4)$. In case the boundary of E_2 is a circle the isoperimetric deficits Δ_i of E_i satisfy the relation $\Delta_1=\Delta_3=\Delta_4$. These results are generalizations of investigations by Jordan and Fiedler, Ganapathi and Vincensini in the case where $E_1=E_2$.

H. Busemann (Northampton, Mass.).

Varela Gil, J. Multiple orbiform curves. Univ. Nac. Tucumán. Revista A. 4, 55–68 (1944). (Spanish) [MF 13019]

The definition of curves of constant width is generalized to include all closed curves with multiple points such that each normal is a double normal, being normal to two points on the curve at a constant distance apart. The author considers certain curvilinear polygons with an odd number of sides, with a cusp of the first type at each vertex, but such that each side is a simple arc without further singularities. Multiple curves of constant width are constructed by describing an involute of such a polygon and relations involving the total curvatures of both sets of curves are given.

D. Derry (Saskatoon, Sask.).

Vázquez García, Roberto. Hypersurfaces with width. Bol. Soc. Mat. Mexicana 2, 1–11 (1945). (Spanish) [MF 12867]

Let $\tau=(p_1,\cdots,p_{n-1},-1)$ define a direction in Euclidean n-space. The author considers differentiable hypersurfaces which have exactly two normals parallel to every direction τ. Let $2r$ be the distance between the two tangent hyperplanes with normal τ. A formula is developed for the coordinates of a point on the hypersurface in terms of r, an analytically defined function $U(p_1,\cdots,p_{n-1})$ and the partial derivatives of these two functions. Four applications of the formula are made, most of which lead either to known results concerning bodies of constant width or to generalizations. [The reviewer would like to point out that in the case in which the surface encloses a convex body its "Stützfunktion" $H(u_1,\cdots,u_n)$ may be defined in terms of U and r and the author's formula is Minkowski's expression for the coordinates $x_\nu=\partial H/\partial u_\nu$, after an appropriate specialization of the direction vector $(u_1,\cdots,u_n)$. Cf. Bonnesen and Fenchel, Theorie der Konvexen Körper, Springer, Berlin, 1934, pp. 25–26.]

D. Derry (Saskatoon, Sask.).

Santaló, L. A. On the circle of maximum radius contained in a domain. Revista Unión Mat. Argentina 10, 155–162 (1945). (Spanish) [MF 12871]

Let R be a domain whose boundary is a closed rectifiable Jordan curve of length L and area F. Let ρ_1 be the least upper bound of the radii of all the circles that are contained in R. Grünwald and Turán published a proof that $\rho_1\geqq(4\pi+2)^{-1}F/L$ and announced that Grünwald and Vázsonyi had proved that (1) $\rho_1\geqq F/L$ [Acta Litt. Sci. Szeged 8, 236–240 (1937)]. The author proves (1) and shows that (1) cannot be replaced by an inequality of the form $\rho_1\geqq cF/L$, where $c>1$.

Notations: $C(P)$ denotes the circle of radius ρ and center P; $n(P)$, the number of points of intersection of the boundaries of P and $C(P)$; $\nu(P)$, the number of simply connected

components of $R\cdot C(P)$; M_0, the measure of the set of the P's with $C(P)\subset R$. Then

$$M_0=\int_{n>0}(\tfrac{1}{2}n-\nu)dP+F+\pi\rho^2-L\rho$$

if no $C(P)$ contains R. Since the integrand is nonnegative, this identity implies

$$M_0\geqq F+\pi\rho^2-L\rho, \tag{2}$$

and (2) is sure to hold if $\pi\rho^2<F$. Choosing $\rho=F/L$, the author obtains (1). [Let $\rho_0=(2\pi)^{-1}\{L-(L^2-4\pi F)^{\frac{1}{2}}\}$; thus $\rho_0>F/L$, $\pi\rho_0^2<F$ and $F+\pi\rho_0^2-L\rho_0=0$. Choosing $\rho=\rho_0-\epsilon$ and letting $\epsilon\rightarrow 0$, the better inequality $\rho_1\geqq\rho_0$ is obtained.] By the same method the corresponding problems are solved in elliptic and hyperbolic geometry ($\tan\rho_1\geqq F/L$ and $\tanh\rho_1\geqq F/L$, respectively). *P. Scherk.*

Santaló, L. A. Origin and development of integral geometry. Revista Univ. Católica Perú 12, 205–230 (1944). (Spanish) [MF 14184]

This is a general account, with an extensive bibliography. [The original article was paged 105–130, but the pagination is corrected on an errata slip.]

Algebraic Geometry

Kasner, Edward. Algebraic curves, symmetries and satellites. Proc. Nat. Acad. Sci. U. S. A. 31, 250–252 (1945). [MF 13291]

Let C be an algebraic curve of order n in the complex projective plane. The author considers the symmetrical correspondence T in which a pair of points P, P' correspond if the intersections (other than the circular points at infinity) of the minimal lines through P and P' lie on C. The correspondence T is, in general, of degree n^2 but may be of lower degree in special cases (for example, if C is a circle, the degree is unity). The image of C under T consists of C and a second curve called the satellite of C. In general, the order of this curve is $n(n-1)^2$, but it may be less.

The author announces a number of theorems concerning such transformations. If C is a conic then any power of T has degree 4. In general, for a curve of order n, the order of T^2 is $n^2(n-1)^2$. If C is given by an equation $\phi(x, y)=0$, where $\phi_{xx}+\phi_{yy}=0$, then the satellite of C coincides with C. *J. A. Todd* (Cambridge, England).

Bottema, O. Eine Bemerkung über die Koppelkurve. Nieuw Arch. Wiskde. 22, 9–14 (1943). [MF 14300]

The theory of Weierstrassian elliptic functions is here applied to the three-bar curve, a tricircular trinodal sextic which is the locus of a point rigidly attached to the link CD of a jointed plane quadrilateral $ABCD$, clamped at A and B. Two points on the cubic $x=\wp'(t)$, $y=\wp(t)$ are said to be in Hessian correspondence if their parameters differ by a half-period. A point-pair Q_1Q_2 is called the Laguerre transform of a given point-pair P_1P_2 if the intersections $(P_1Q_1\cdot P_2Q_2)$ and $(P_1Q_2\cdot P_2Q_1)$ are the circular points at infinity. If P_1, P_2 are in Hessian correspondence, the locus of Q_1 (or Q_2) is a three-bar curve. The author ascribes this result to Weiss [Math. Z. 47, 187–198 (1941); these Rev. 4, 51]. He goes on to show that, if two of the three points at infinity on the cubic are paired in the Hessian correspondence, then the sextic consists partly of the line at infinity taken twice, so that the rest of it is a circular quartic. *H. S. M. Coxeter* (Toronto, Ont.).

Wachs, Sylvain. Sur une propriété arithmétique d'une variété cubique de l'espace à quatre dimensions. Rev. Sci. (Rev. Rose Illus.) 80, 402–406 (1942). [MF 13989]

This paper consists of two unrelated parts. In the first part it is observed that on the general V_3^3 in [4] there are sets of three lines, two by two skew; then, by a simple construction involving such a set of three lines, the well-known result that V_3^3 is the image of an involution of order 2 in [3] is proved. [This result can also be obtained by using a single line r of V_3^3, from the remark that the lines touching V_3^3 at points of r form an ∞^3 unicursal system, whose elements are in an algebraic (2, 1)-correspondence with the points of V_3^3.] When V_3^3 and the three lines considered on it belong to the rational field, one thus obtains a rational 3-parameter solution for the Diophantine equation representing V_3^3. The author asks whether this solution is complete or not, without giving an answer. [The answer is, in general, negative, since the conic Γ considered in §2 may contain an infinity of rational points, and yet meet the line D at two irrational points.]

The second part determines all the rational lines lying on the V_3^3 (1) $x_0^3+x_1^3+x_2^3+x_3^3+x_4^3=0$, that is, all the solutions of (1) which are of the form (2) $x_i=a_ix_3+b_ix_4$ (a_i, b_i rational, $i=0, 1, 2$). The author obtains them by discussing the system of four Diophantine equations in a_0, a_1, a_2, b_0, b_1, b_2 expressing that the elimination of x_0, x_1, x_2 from (1) and (2) leads to an identity in x_3, x_4. The result is that there are only 15 such solutions, namely, the trivial ones $x_i+x_j=x_h+x_k=x_l=0$, where i, j, h, k, l are the numbers 0, 1, 2, 3, 4 taken in any order; but the author twice states erroneously that there are only the 10 solutions $x_i+x_j=x_h=x_k=x_l=0$. *B. Segre* (Manchester).

Wiman, A. Über Regelflächen von beliebig hohem Grade mit vollständig zerfallenden Doppelkurven. Acta Math. 76, 1–30 (1945). [MF 13197]

A plane π through a generator l of a ruled surface R of degree n in three-dimensional complex projective space intersects R, in addition to l, in a curve C of degree $n-1$. Hence C intersects l in $n-1$ points, one of which is the point of contact of π with R. The remaining $n-2$ points belong to the double curve D of R, so that another generator of R passes through each of these points. This defines an $(n-2, n-2)$-correspondence K between the generators. This paper determines those R for which D is completely reducible. In that case K consists of $n-2$ one-to-one correspondences, and the study of this case forms the bulk of the paper. There are too many cases to be enumerated here. It must suffice to state that all R belong to the following types. (1) The generators belong to the congruence of the bisecants of a rational W-curve. (2) The generators all intersect a fixed straight line. (3) The generators all meet a fixed conic section E and are tangent to a cone of degree two which contains E. *H. Busemann.*

Segre, B. The biaxial surfaces, and the equivalence of binary forms. Proc. Cambridge Philos. Soc. 41, 187–209 (1945). [MF 13419]

Two binary forms $f(x, y)$ and $f_1(x_1, y_1)$, of the same degree n, are equivalent if there is a transformation $x_1=\alpha x+\beta y$, $y_1=\gamma x+\delta y$, with $\alpha\delta-\beta\gamma\neq 0$, carrying f_1 into f. The exis-

tence of such a transformation is equivalent to the existence, on the surface $F=f(x,y)-f_1(x_1,y_1)=0$ in projective 3-space, of a line not intersecting the two lines $x=y=0$, $x_1=y_1=0$; F is called a biaxial surface, the two lines being the axes. In this paper equivalence over the complex field is investigated in the case in which neither f nor f_1 has a multiple factor; F is then nonsingular. Let f, f_1 have Hessians h, h_1 and discriminants D, D_1. Then for $n>2$ the lines on F not meeting the two axes can be characterized in either of the following ways. (a) The linear components of the intersection of F and $H_\lambda\equiv\lambda h(x,y)-h_1(x_1,y_1)=0$, where λ has one of the values $(D_1/D)^{2/n(n-1)}$; (b) the components of the flecnodal curve of F which lie on H_λ for some λ. From this it follows that the required lines are the curves of intersection of F, H_λ, and $\lambda^3k(x,y)-k_1(x_1,y_1)=0$, k being a covariant of f of order $4n-12$, for the values of λ specified in (a). The special cases $n=3$ and $n=4$ are considered in detail, and the results shown to agree with those already known. *R. J. Walker* (Ithaca, N. Y.).

Lesieur, Léonce. Tangentes principales d'une variété à p dimensions dans l'espace à n dimensions. C. R. Acad. Sci. Paris **220**, 724–726 (1945). [MF 14065]

Let V_p be a (differentiable) variety of dimension p lying in a linear space of n dimensions. Then if $2p<n$ the tangent p-spaces to V_p generate a variety of dimension $2p$ which has the same tangent space at all points of any tangent to V_p. If $2p\geqq n$, at each point of V_p there are ∞^{n-p-2} "principal tangents" of V_p whose locus is a variety of dimension $n-1$ having the same tangent hyperplane at all points of a principal tangent to V_p. The principal tangents at any point of V_p form an algebraic cone of order n^p-p-1, the locus of double lines of a linear system of dimension $n-p-1$ of quadric cones, which are the cones of tangents of the sections of varieties containing V_p by the tangent space to V_p at the point. *J. A. Todd.*

Apéry, Roger. Sur certains caractères numériques d'un idéal sans composant impropre. C. R. Acad. Sci. Paris **220**, 234–236 (1945). [MF 13497]

In the projective space $[n]$, let $\mathfrak{m}$ be an ideal without improper components; let φ_1, φ_2 be two hypersurfaces of orders k_1, k_2 not passing through an essential variety of $\mathfrak{m}$; let $\mathfrak{a}_1$, $\mathfrak{a}_2$ be the ideals of $(\mathfrak{m},\varphi_1)$, $(\mathfrak{m},\varphi_2)$; and let α_1, α_2 be the numbers of linearly independent hypersurfaces of orders $1+k_1$, $1+k_2$, mod $\mathfrak{a}_1$, mod $\mathfrak{a}_2$, the product of which with any hypersurface belongs to $\mathfrak{a}_1$, $\mathfrak{a}_2$, respectively. Under these circumstances the author proves that $\alpha_1=\alpha_2$. As an application it is proved that a necessary and sufficient condition that an irreducible variety without singularities be of the first kind is that the hypersurfaces of any order cut on V a complete linear series. If V is a curve of $[3]$, of order m, genus p, then $p>lm-C^3_{l+3}$ for every l. This inequality is necessary but not sufficient: if $m=9$, $p=10$, two different curves of different kinds exist. *V. Snyder.*

Apéry, Roger. Sur les courbes de première espèce de l'espace à trois dimensions. C. R. Acad. Sci. Paris **220**, 271–272 (1945). [MF 13500]

[Cf. the preceding review.] The author now shows that in [3] every curve of the first kind is a complete intersection. The method of proof does not apply to varieties in space of more dimensions. *V. Snyder.*

Gaspar, Eduardo. On the representation of rational normal varieties. Publ. Inst. Mat. Univ. Nac. Litoral **5**, 269–281 (1945). (Spanish) [MF 13733]

A rational normal variety is defined by homogeneous polynomials ξ_j and homogeneous point coordinates x_i defined by the equations

$$x_i=\prod_{j=1}^{r}\xi_j^{q_j(i)},\qquad \sum_{j=1}^{r}q_j(i)=n.$$

The variety thus defined is birationally representable in $[r-1]$ defined parametrically by the ξ_j. The purpose of the note is the characterization of a minimum base for an ideal in those polynomials which represent the varieties of $[z-1]$ which pass through the normal variety defined above. The problem was first considered by D. Hilbert [Math. Ann. **36**, 473–534 (1890)], who treated in detail the case $r=2$, $n=3$. It was taken up later by A. Hurwitz [Math. Ann. **79**, 313–320 (1919)], who considered a number of quadratic varieties which appear as second minors of a matrix, the elements of which are $\sum\xi_i\varphi_i$; each φ_i represents the totality of distinct monomials of order $n-1$ in the parameters ξ_j. The note completes the problem proposed by Hurwitz. *V. Snyder* (Ithaca, N. Y.).

Godeaux, Lucien. Un problème sur les variétés algébriques. Revue Sci. (Rev. Rose Illus.) **80**, 6–9 (1942). [MF 13824]

The author first recalls that for algebraic curves a necessary and sufficient condition that a curve is rational is that its genus is zero. In the case of surfaces, the bigenus is introduced. A necessary and sufficient condition that an algebraic surface is rational is that its arithmetic genus p_a and its bigenus P_2 each are zero. Another surface cited is the sextic F_6 of Enriques, having the six edges of a tetrahedron for double lines. This has the added property that the complete system of its plane sections is its own adjoint; a canonical curve does not exist but the second adjoint of the plane sections does exist and is of order zero. This surface is the image of an involution of order two, having no fixed points, belonging to a surface for which $p_a=P_4=1$ and such that each linear system on the surface is its own adjoint. The present note generalizes these ideas to apply to algebraic varieties of more than two dimensions with the following properties: (a) each system of linear varieties of $r-1$ dimensions is its own adjoint; (b) the characteristic of each such system is complete; (c) on each variety of $r-1$ dimensions on V_r the adjoint of $r-2$ dimensions cuts a complete system. Various results are given. *V. Snyder* (Ithaca, N. Y.).

Godeaux, Lucien. Construction d'une surface algébrique irrégulière. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 408–422 (1943). [MF 13851]

Godeaux, Lucien. Sur la construction d'une surface d'irrégularité trois. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30**, 815–822 (1943). [MF 13868]

Godeaux, Lucien. Sur certaines involutions appartenant à une surface algébrique et n'ayant que des points unis parfaits. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 495–502 (1943). [MF 13853]

Godeaux, Lucien. Sur une involution du second ordre appartenant à une surface d'Enriques. Bull. Soc. Roy. Sci. Liége **13**, 188–191 (1944). [MF 13174]

Godeaux, Lucien. Sur les involutions cycliques appartenant à une variété algébrique de genres un. Bull. Soc. Roy. Sci. Liége 13, 277–288 (1944). [MF 13181]

Godeaux, Lucien. Sur les involutions régulières du second ordre appartenant à une surface irrégulière. Bull. Soc. Roy. Sci. Liége 14, 2–10 (1945). [MF 13187]

Godeaux, Lucien. Sur une propriété des surfaces de genres un et de rang deux. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 622–636 (1943). [MF 13858]

Godeaux, Lucien. Sur les surfaces de Kummer généralisées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 724–735 (1943). [MF 13865]

Godeaux, Lucien. Sur les surfaces inscrites dans une surface de Kummer. Bull. Soc. Roy. Sci. Liége 13, 128–136 (1944). [MF 13169]

Godeaux, Lucien. Sur le contact des surfaces cubiques. Bull. Soc. Roy. Sci. Liége 13, 2–10 (1944). [MF 13159]

Godeaux, Lucien. Sur le contact de surfaces le long de courbes. Bull. Soc. Roy. Sci. Liége 13, 46–58 (1944). [MF 13162]

Godeaux, Lucien. Sur un théorème de Salmon concernant les droites d'une surface cubique. Bull. Soc. Roy. Sci. Liége 13, 58–62 (1944). [MF 13163]

Godeaux, Lucien. Construction d'une surface canonique du septième ordre. Bull. Soc. Roy. Sci. Liége 13, 94–97 (1944). [MF 13167]

Godeaux, Lucien. Sur des variétés algébriques possédant une hypersurface canonique ou bicanonique d'ordre zéro. Bull. Soc. Roy. Sci. Liége 13, 227–232 (1944). [MF 13178]

Godeaux, Lucien. Sur une surface du quatrième ordre possédant dix points doubles. Bull. Soc. Roy. Sci. Liége 13, 271–277 (1944). [MF 13180]

Godeaux, Lucien. Sur une propriété des surfaces d'ordre n circonscrites à un $(n+1)$-èdre. Bull. Soc. Roy. Sci. Liége 13, 301–306 (1944). [MF 13184]

Godeaux, Lucien. Sur la surface représentant les couples de points d'une courbe de genre trois. Bull. Soc. Roy. Sci. Liége 13, 311–323 (1944). [MF 13185]

Baudoux, Roger. Sur les involutions du second ordre de l'espace dont les couples appartiennent aux rayons d'un complexe linéaire. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 845–858 (1942). [MF 13682]

Baudoux, Roger. Sur les involutions du second ordre de l'espace dont les couples appartiennent aux rayons d'un complexe linéaire. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 8–14 (1943). [MF 13835]

Baudoux, Roger. Sur les involutions du second ordre de l'espace dont les couples appartiennent aux rayons d'un complexe linéaire. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 321–342 (1943). [MF 13869]

Burniat, Pol. Sur les systèmes i-canoniques des surfaces cycliques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 204–214 (1943). [MF 13845]

Burniat, Pol. Sur des surfaces sous-canoniques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 645–652 (1943). [MF 13860]

Burniat, Pol. Sur une surface canonique de l'espace S_4. Bull. Soc. Roy. Sci. Liége 14, 10–18 (1945). [MF 13188]

D'Orgeval, B. Les plans multiples représentatifs d'une surface algébrique et la méthode de M. Chisini. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 215–229, 653–656 (1943). [MF 13846]

Gauthier, Luc. Rationnalité de l'intersection de p hyperquadriques dans l'espace à r dimensions. Bull. Soc. Roy. Sci. Liége 13, 191–195 (1944). [MF 13175]

Jongmans, F. Les variétés algébriques à trois dimensions dont les courbes-sections ont le genre trois. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30, 766–782, 823–835 (1943). [MF 13867]

Lorent, H. Sur les points associés d'une biquadratique gauche de première espèce. Bull. Soc. Roy. Sci. Liége 12, 599–601 (1943); 13, 307–310 (1944). [MF 13155]

Nollet, Louis. Sur les congruences linéaires de courbes unicursales et la détermination des congruences linéaires de cubiques gauches. Bull. Soc. Roy. Sci. Liége 13, 98–110 (1944). [MF 13168]

Nollet, L. Sur un faisceau de surfaces du neuvième ordre à sections elliptiques. Bull. Soc. Roy. Sci. Liége 13, 323–325 (1944). [MF 13186]

Pire, N. Sur les surfaces ayant une conique et une droite multiples. Bull. Soc. Roy. Sci. Liége 13, 84–91 (1944). [MF 13166]

Pire, N. Sur la construction de quelques systèmes linéaires surabondants de courbes planes. Bull. Soc. Roy. Sci. Liége 13, 158–162 (1944). [MF 13173]

Pire, N. Sur la représentation plane de certaines surfaces rationnelles. Bull. Soc. Roy. Sci. Liége 13, 195–203 (1944). [MF 13176]

Pissard, N. Sur la génération de quelques courbes algébriques. Bull. Soc. Roy. Sci. Liége 13, 288–293 (1944). [MF 13182]

Gérard, J. Sur certains systèmes linéaires surabondants de courbes planes. Bull. Soc. Roy. Sci. Liége 13, 16–19 (1944). [MF 13161]

Gérard, J. Sur une surface du septième ordre possédant deux cubiques gauches doubles et sur certains systèmes linéaires surabondants de courbes planes. Bull. Soc. Roy. Sci. Liége 13, 151–158 (1944). [MF 13172]

Gérard, J. Sur une transformation birationnelle de l'espace. Bull. Soc. Roy. Sci. Liége 13, 74–84 (1944). [MF 13165]

Keller, Ott-Heinrich. Zu einem Satze von H. W. E. Jung über ganze birationale Transformationen der Ebene. J. Reine Angew. Math. 186, 78–79 (1944). [MF 13404]

The author presents a brief proof that each integral birational transformation of the plane is a product of affinities and transformations like $x'=x$, $y'=y+ax^n$.

O. F. G. Schilling (Chicago, Ill.).

Apéry, Roger. Action de certaines transformations birationnelles sur les nombres de Betti des variétés algébriques. C. R. Acad. Sci. Paris 217, 435–437 (1943). [MF 11663]

Let $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ be nonsingular algebraic varieties of dimension n, which are related by a birational correspondence which is one-to-one except for a d-dimensional variety $V \subset \mathfrak{B}$ whose inverse image $\bar{V} \subset \bar{\mathfrak{B}}$ is homeomorphic with the product of V with a projective space S of dimension $n-d-1$. Let $R_i, \bar{R}_i, \rho_i, \bar{\rho}_i$ be the Betti numbers of $\mathfrak{B}, \bar{\mathfrak{B}}, V, \bar{V}$, respectively. Then $R_i-\rho_i=\bar{R}_i-\bar{\rho}_i$. The author states that analogous results can be obtained if V is replaced by the union of disjoint subvarieties of $\mathfrak{B}$, as well as when $\bar{\mathfrak{B}}$ is allowed to contain fundamental varieties.

G. W. Whitehead (Princeton, N. J.).

Calvo, D. Représentation d'une transformation birationnelle de Caporali. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 657–665 (1943). [MF 13861]

Calvo, D. Sur les transformations birationnelles de l'espace. Bull. Soc. Roy. Sci. Liége 13, 62–73 (1944). [MF 13164]

Derwidué, L. Sur les éléments fondamentaux des transformations birationnelles hyperspatiales. Mém. Soc. Roy. Sci. Liége (4) 4, 1–17 (1941). [MF 14012]

Derwidué, L. Sur les transformations birationnelles de l'espace laissant invariantes les courbes d'une congruence linéaire. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 187–193 (1943). [MF 13843]

Differential Geometry

Santaló, L. A. Surfaces whose D-curves are geodesics or isogonal trajectories of the lines of curvature. Publ. Inst. Mat. Univ. Nac. Litoral 5, 255–267 (1945). (Spanish) [MF 13732]

A D-curve on a surface S in E^3 is a curve whose osculating sphere is everywhere tangent to the tangent plane of S. The only surfaces whose geodesics are D-curves are the sphere and the (circular) cylinder. If the D-curves of a surface cut the lines of curvature at constant angles then the surface is a (circular) cone, a cylinder, a torus or a cyclide of Dupin. The only surfaces whose geodesics have constant torsion are the cylinder, the sphere and the plane.

H. Busemann (Northampton, Mass.).

Kasner, Edward, and DeCicco, John. Converse of Ptolemy's theorem on stereographic projection. Proc. Nat. Acad. Sci. U. S. A. 31, 338–342 (1945). [MF 13718]

The authors prove that the sphere is the only surface in 3-space for which there exists a conformal perspectivity from a fixed point to a fixed plane. *S. B. Myers.*

Kasner, Edward, and DeCicco, John. A new characteristic property of minimal surfaces. Bull. Amer. Math. Soc. 51, 692–699 (1945). [MF 13606]

The present paper is a sequel to an earlier one [Proc. Nat. Acad. Sci. U. S. A. 31, 44–50 (1945); these Rev. 6, 186]. The authors have applied their extension of Lie's theorem to obtain results of which the following is typical. If a surface Σ possesses more than four distinct isothermal systems of parallel plane sections, all the planes of which are parallel to a given rectilinear direction, then Σ is either a minimal surface or a sphere, and therefore every system of parallel plane sections is isothermal. As the authors point out, this shows that the well-known property of minimal and spherical surfaces, namely, that every system of parallel plane sections is isothermal, is indeed a characteristic property of those surfaces. *M. O. Reade.*

Beckenbach, E. F. Painlevé's theorem and the analytic prolongation of minimal surfaces. Ann. of Math. (2) 46, 667–673 (1945). [MF 14126]

Results concerning the analytic prolongation of a minimal surface under certain boundary conditions have been given by Douglas [Proc. Nat. Acad. Sci. U. S. A. 25, 375–377 (1939); these Rev. 1, 244] and the author [Duke Math. J. 9, 109–111 (1942); these Rev. 3, 309]. The author now generalizes to minimal surfaces Painlevé's classic theorem on the analytic continuation of holomorphic functions of a complex variable. Let finite simply-connected domains D_p, $p=1, 2$, abut along the rectifiable arc C; let the functions (1) $x_j=x_{j,p}(u, v)$, $j=1, 2, 3$, be the coordinate functions of a minimal surface S_p in isothermic representation for (u, v) in D_p, $p=1, 2$, and let (2) $X_j=X_{j,p}(u, v)$, $j=1, 2, 3$, be the direction-cosines of the normals to S_p. Then necessary and sufficient conditions that the functions $x_{j,1}(u, v)$ and $x_{j,2}(u, v)$ be analytic prolongations of each other across C, for $j=1, 2, 3$, are that the functions (1) and (2) converge uniformly on each closed subset of C and that on C the continuous boundary functions thus defined satisfy

$$x_{j,1}(u, v)=x_{j,2}(u, v), \quad X_{j,1}(u, v)=X_{j,2}(u, v), \quad j=1, 2, 3.$$

In his proof the author makes essential use of results in the differential geometry of minimal surfaces, such as the classic Weierstrass theorem on the isothermic representation of minimal surfaces [T. Radó, On the Problem of Plateau, Springer, Berlin, 1933, p. 27], and of results in the theory of analytic functions of a complex variable, such as the theorem of Painlevé referred to above.

M. O. Reade (Lafayette, Ind.).

Blaschke, Wilhelm. Zur Bewegungsgeometrie auf der Kugel. Comment. Math. Helv. 17, 80–82 (1945).

Steiner's "römische Fläche" is derived with the help of kinematical concepts. Consider a closed 1-parameter family of rotations of 3-space around the origin and define $v_x=\int x\times dx$, where the integral is taken over the curve which the vector x describes under the family of rotations. If x runs through the unit sphere, v_x describes the "römische Fläche." *H. Samelson* (Syracuse, N. Y.).

Blaschke, Wilhelm. Über flächenläufige Bewegungsvorgänge. Comment. Math. Helv. 17, 278–282 (1945).

Some formulae for 2-parameter families of motions of 3-space are given in terms of Study's formalism of "dual" vectors. An application is made to the normal lines of the line congruences generated by such families of motions; a normal line in a congruence is one at which the focal planes are perpendicular. *H. Samelson* (Syracuse, N. Y.).

Springer, C. E. Union curves and union curvature. Bull. Amer. Math. Soc. 51, 686–691 (1945). [MF 13605]

Let S be a surface in three-dimensional Euclidean space. A union curve C on S relative to a given rectilinear congruence R is defined as the curve whose osculating plane

at each point of C contains the line of R through the point. The author then defines the union curvature of any curve C' on S relative to R in a manner exactly analogous to the definition of the geodesic curvature of C' except that the direction of projection of C' onto the tangent plane of S coincides with that of the corresponding line of R. It is proved that the union curves are the curves of zero union curvature. *A. Fialkow* (New York, N. Y.).

Llensa, Georges. Sur les propriétés de dérivabilité relatives à certains systèmes triples orthogonaux. C. R. Acad. Sci. Paris **220**, 297–298 (1945). [MF 13943]

The author has previously [Bull. Sci. Math. (2) **65**, 225–250 (1941); these Rev. **7**, 77] studied the partial differential equation (*) $(\overline{MO}^2-R^2)\cdot\operatorname{grad}^2 u=1$, where M is the point (x, y, z) and $O(u)$ and $R(u)$ refer to the center and radius of a given family $S(u)$ of spheres; the equation (*) appears in the theory of triply orthogonal systems of surfaces. He now states the condition that a single surface Σ_{u_0}, disjoint from the sphere $S(u_0)$, must satisfy in order to belong to a solution of (*), in the sense of "intégrale paratingente" defined in the paper; the condition is that the curvatures of Σ_{u_0} are bounded. Similarly, the condition that Σ_{u_0} belongs to a triply orthogonal system of solutions of (*) is that it possesses continuous second derivatives.

H. Samelson (Syracuse, N. Y.).

Delgleize, A. Sur la représentation conforme des surfaces. Bull. Soc. Roy. Sci. Liége **12**, 353–372 (1943). [MF 13147]

The author has studied the point correspondences between two surfaces (S) and (Σ) such that the null lines of (Σ) correspond to the asymptotic lines of (S), the points of (Σ) lying in the tangent planes of (S). This class of surfaces (S) contains as special cases classes studied by Darboux and Weingarten and other classes studied by Bianchi. All such surfaces (Σ) are in conformal representation. If the parametric curves of (S) are conjugate, then those of (Σ) are orthogonal, and conversely. Applications are made to subclasses of surfaces with interesting properties. Some of these are as follows. (1) Surfaces (S) such that the tangents to ∞^1 geodesics are normal to a surface (Σ). All such (Σ) are conformally equivalent, the corresponding points lying in the tangent lines of the given family of geodesics. (2) There exist ∞^2 surfaces (S) which are conformally represented on two minimal surfaces (Σ) and (Σ') (of Ribaucour) such that corresponding points lie on the lines $S\Sigma$ and $S\Sigma'$, the normals to these surfaces belonging to the tangent plane of (S). (3) Surfaces (S) where one can associate ∞^2 surfaces (Σ) conformally represented on a minimal surface. The corresponding points of the surfaces (Σ) are such that the normals at these points are in the tangent plane of (S). This is connected with a problem of Bianchi. (4) There exists an infinitude of surfaces (S) conformally represented on two surfaces (Σ) and (Σ') of constant mean curvature. Other applications are given concerning mappings on minimal suriaces and on surfaces of constant mean curvature. *J. DeCicco.*

Roşca, R. Sur les réseaux (M). Bull. École Polytech. Bucarest [Bul. Politehn. Bucureşti] **13**, 295–303 (1942). [MF 13567]

This paper contains the following three theorems concerning conjugate (M) nets: (1) Let (x) be an (M) net and let (x') (also an (M) net) be any Moutard transform of (x). Let (β) and (β') be the reciprocal polars of (x) and (x'), respectively, with respect to the fundamental quadric Q. The focal nets (η), (η') of the congruence of lines $(\beta\beta')$ and the focal nets (z), (z') of the congruence (xx') are described by the midpoints of the segments $(\beta\beta')$ and (xx'), respectively. In the metric established by Q the segments (xx'), $(\beta\beta')$ are equal. These four focal nets (η), (η'), (z), (z') in proper order form a periodic sequence of Laplace. (2) Let (x') and (x'') be two (M) nets related to an (M) net (x) by a transformation of Moutard. Among the ∞^1 conjugate nets related to (x') and (x'') by a transformation of Moutard, there is only one other besides (x) which is an (M) net. (3) Let two (M) nets be related by a transformation of Moutard, and let their sustaining surfaces be surfaces of Clifford. Then the conjugate congruence of these nets is an isotropic-normal congruence. *V. G. Grove.*

Mihaïlescu, Tiberiu. Les réseaux conjugués à transformés de Laplace en correspondance asymptotique. Bull. Math. Phys. Éc. Polytech. Bucarest **11**, 40–139 (1940). [MF 13548]

The object of this memoir is the study of special classes of conjugate nets in a projective space of three dimensions. The method is that of Cartan, using the Pfaffian forms and equations characteristic of that method. [Some of the references are incorrect; in particular, credit is given to Green for the complete geometrical characterization of isothermally conjugate nets rather than to Wilczynski [Amer. J. Math. **42**, 211–221 (1920)].] The chapter headings and characteristic theorems for each chapter follow.

(1) Fundamental formulas. (2) Nets whose Laplace transforms are in asymptotic correspondence. On an arbitrary surface there exist conjugate nets whose Laplace transforms have their asymptotic curves in correspondence; such nets depend on two arbitrary functions of one variable. (3) R nets. A new proof is given of Demoulin's theorem that a surface sustaining an R net sustains an infinity of them. The only ruled surfaces sustaining an R net are those with rectilinear flecnode curves; on such surfaces the R nets depend on an arbitrary function of one variable. The most general surface such that the Laplace transforms of one of its R nets are quadrics is a ruled surface (with rectilinear flecnode curves).

(4) Conjugate nets of Terracini-Pantazi (TP nets). A TP conjugate net is defined as follows. Let Γ be a congruence of lines l and let x, y be the focal points on l; let the focal planes of a second line l' of Γ intersect l in points x', y'. The limiting value of one of the cross ratios of x, y, x', y' is called the projective linear element (at l) of Γ. Two congruences are said to be projectively applicable of the second order if their lines are in one-to-one correspondence and their projective linear elements are identical at corresponding lines. A conjugate net is a TP net if the congruences of its tangent lines are projectively applicable of the second order; TP nets have equal corresponding point and tangential invariants, and are R nets. The ray and axis congruences of a TP net are stratified in the order ray to axis. If a conjugate net composed of the curves of Darboux-Segre has ray and axis congruences stratified, ray to axis, then it is a TP net.

(5) Isothermally conjugate nets. Isothermally conjugate nets on whose Laplace transforms the asymptotic curves correspond are either R nets or nets whose anti-ray and axis curves coincide. These latter are called D nets. The D nets lying on a ruled surface are of two types: (a) R nets,

the surface having rectilinear flecnode curves; (b) nets with equal point and tangential invariants, the surface having distinct rectilinear flecnode curves. In each type the surface sustains an infinity of D nets depending on two arbitrary constants. The theorems are proved only for nonharmonic isothermally conjugate nets.

(6) Conjugate nets of Darboux-Segre curves whose Laplace transforms are in asymptotic correspondence. If a congruence of tangent lines to a surface S_1 has the asymptotic curves on the second focal surface S_2 corresponding to a net on S_1 composed of curves of Darboux, the correspondence between S_1 and S_2 is said to be the correspondence E. The correspondence established between S_1 and S_2 by the congruence of tangents to one of the families of curves of Darboux (or Segre) is the correspondence E, and with the exception of congruences one of whose focal surfaces is quadric, this is the only manner of establishing the correspondence. On any surface there exist two conjugate nets of Darboux-Segre curves whose Laplace transforms are in asymptotic correspondence, and the congruences of whose tangents establish the correspondence E between their focal surfaces; the third net formed by the other Darboux-Segre curves is a TP net.

(7) Conjugate nets whose Laplace transforms are projectively applicable. Nets whose Laplace transforms are projectively applicable are either R or D nets.

V. G. Grove (East Lansing, Mich.).

Terracini, Alejandro. The varieties of Grassmann and partial differential equations of first order in the case of several independent variables. Univ. Nac. Tucumán. Revista A. 4, 363–376 (1944). (Spanish) [MF 13033]

The author presents a geometric interpretation of the characteristics of a partial differential equation of first order in a single unknown function of k independent variables. In the case $k=2$, any line of ordinary space T_3 is mapped by use of Pluecker coordinates into a point of the Klein four-dimensional quadric M_4 of projective five-space S_5. A first order surface element E_1 of T_3 is considered to be composed of the tangent lines at a given point of a surface in T_3, and hence is represented by a line g in M_4. Thus a first order partial differential equation in one unknown function is depicted in M_4 by a system Γ of ∞^4 lines. The points P_1 and P_2 on any nonspecial line g are said to be fundamental if they are also on the infinitely near lines of Γ which lie on the two planes π_1 and π_2 of M_4 which pass through g. The characteristics are pictured by the ∞^3 ruled surfaces R, whose rulings all belong to Γ, such that the tangent planes at the fundamental points P_1 and P_2 of any generator g of R coincide with the planes π_1 and π_2 of M_4.

To extend these results to the general case $k>2$ is a much more difficult problem. The procedure is to map the lines of a $(k+1)$-dimensional space T_{k+1} by means of Pluecker coordinates into points of a Grassmann $2k$-dimensional variety G_{2k} of a projective n-space S_n, where $n=k(k+3)/2$. The order of G_{2k} is $(1/k)\binom{2k}{k-1}$. A first order partial differential equation in one unknown function is represented in G_{2k} by a system Γ of ∞^{2k} $(k-1)$-dimensional spaces S_{k-1}. By introducing numerous other concepts of more complicated nature than those for the case $k=2$, the author succeeds in giving an elegant geometric interpretation of the characteristics. *J. DeCicco* (Chicago, Ill.).

Terracini, Alejandro. New geometric points of view connected with surface elements and partial differential equations of 2d order. I. Univ. Nac. Tucumán. Revista A. 4, 259–316 (1944). (Spanish) [MF 13029]

The author presents a projective differential geometric study of any given partial differential equation of second order in a single unknown function of two independent variables. Numerous properties of a variety associated with such an equation are obtained.

A line of ordinary space S_3 is mapped by Pluecker coordinates into a point of the four-dimensional quadric M_4 of Klein of a projective five-space S_5. A first order surface element E_1 is represented by a line g in M_4. Thus a partial differential equation of first order is pictured by a system Γ of ∞^4 lines in M_4. A nonparabolic second order surface E_2 of S_3 is depicted in M_4 by a bi-pencil, defined in the following manner. The basic E_1 of E_2 is mapped into a line g in M_4. The images of the two asymptotic lines of E_2 are two points F_1 and F_2 on g. As E_2 varies over a surface of S_3, F_1 and F_2 describe two surfaces. Construct the two tangent pencils θ_1 and θ_2 of lines of these at F_1 and F_2. The base planes of these two pencils are correspondents under a fundamental projective involution between the planes through g and contained in the three-space g_2' polar to g with respect to M_4. Thus any bi-pencil of M_4 consists of two pencils (F_1, θ_1) and (F_2, θ_2) in the special geometric position described above. A partial differential equation of second order is pictured by the geometric configuration E of ∞^7 bi-pencils.

The author studies the invariants between two bi-pencils in M_4. These correspond to the invariant of Mehmke for two tangent E_2's, and the two invariants I_1 and I_2 of C. L. Bouton for two nontangent E_2's. A geometric interpretation of each is given in M_4 in terms of cross-ratio. The differential forms of these invariants lead to the introduction of varieties W_i of ∞^i bi-pencils of M_4. Of special interest is the differential form J of I_2. This is a fraction with denominator linear and numerator Ω quadratic in the differentials. For a W_7 (the system E), the study of the degree of the discriminant of Ω leads to particular classes of systems E^* (degree less than 6) and E^{**} (degree less than 5). Geometric characterizations of these special classes are obtained. Next the author studies varieties W_1 along which Ω is zero. Particular attention is directed to the notion of special and fundamental varieties W_1. Various other results in M_4 correspond to the theory of ruled surfaces in ordinary space. The author also considers the question of the geometric representation in the configuration E of the characteristics of a partial differential equation of second order.

J. DeCicco (Chicago, Ill.).

Terracini, Alejandro. On the Monge-Ampère differential equations. Publ. Inst. Mat. Univ. Nac. Litoral 5, 175–198 (1945). (Spanish) [MF 13729]

The author presents a geometric interpretation of a partial differential equation of second order of the Monge-Ampère type. Using Pluecker coordinates, a line of ordinary space S_3 is represented as a point of the four-dimensional Klein quadric M_4 of a projective five-space S_5. A first order surface element E_1 of S_3 is defined as a pencil of tangent lines, and hence is represented by a line g in M_4. A nonparabolic second order surface element E_2 has a more complicated structure in M_4. The asymptotic lines of a surface in S_3 correspond to two surfaces in M_4. Hence any E_2 is represented in M_4 by a line g and the two tangent pencils

of lines θ_1 and θ_2 of these two surfaces constructed at the two points F_1 and F_2, which are on the line g. The base planes of θ_1 and θ_2 are correspondents under a fundamental projective involution j between the planes through g and contained in the three-space g_3' polar to g with respect to M_4. Thus any E_2 is determined completely by g, F_1, F_2, θ_1 and θ_2.

A Monge-Ampère equation of second order is characterized by the fact that, for all the E_2's containing a fixed E_1, there is established in M_4 a rational involutorial quadric correspondence between the pencils (F_1, θ_1) and (F_2, θ_2) such that each pencil whose base plane is a double plane of j corresponds to itself. Another interpretation is obtained by mapping the ∞^2 pencils (F_1, θ_1) into the points of a quadric of ordinary space.

The final result is that, if one forms from the lines of the Klein quadric M_4 a ruled surface R such that, for each point P of a generator g of R, the pencil of lines tangent to R at P possesses a conjugate pencil tangent to R, then these ruled surfaces represent the characteristics of the Monge-Ampère equation. *J. DeCicco* (Chicago, Ill.).

De Cicco, John. Circle-to-line transformations. Amer. Math. Monthly **52**, 425–433 (1945). [MF 13756]

The following theorem, stated by Kasner, is proved: the only systems of ∞^2 circles defined by differential equations of the cubic type

$$y''=A(x, y)+B(x, y)y'+C(x, y)y'^2+D(x, y)y'^3$$

are the linear systems. By means of this theorem, the author derives the eleven parameter group of point transformations $(x, y)\rightarrow(X, Y)$ such that any straight line in the (X, Y) plane corresponds to a circle in the (x, y) plane. Some properties of these circle-to-line transformations are obtained. *A. Fialkow* (New York, N. Y.).

Abramescu, N. Sur la courbure affine de la développée affine d'une courbe plane. Mathematica, Timişoara **21**, 12–18 (1945). [MF 13971]

The affine curvature k of the affine evolute of a plane curve C is derived as an algebraic function of the radius of curvature of C and its first three derivatives with respect to affine arc. The author seeks all curves C for which k is constant or zero, that is, all curves whose evolutes are conics. The case $k=0$ (parabola) leads to the solution of a Riccati equation and the solution is given explicitly in terms of Bessel functions. The case $k=$constant (ellipse or hyperbola) leads to a certain second order differential equation of non-elementary form. *J. L. Vanderslice* (College Park, Md.).

Fialkow, Aaron. Conformal classes of surfaces. Amer. J. Math. **67**, 583–616 (1945). [MF 13941]

In previous papers [Trans. Amer. Math. Soc. **51**, 435–501 (1942); **56**, 309–433 (1944); these Rev. **3**, 307; **6**, 105] the author studied those properties of a subspace V_n $(1\leqq n\leqq m-1)$ imbedded in an enveloping space V_m which remain unchanged under conformal transformations of V_m. These results are applied to the study of a surface V_2 imbedded in a conformal Euclidean space R_3. Some of the main results are as follows. The lines of curvature of a V_2 form an equiareal net relative to the element of conformal area if and only if they coincide with the lines of deviation and hence if and only if V_2 is a gradient surface. Analytic characterizations are obtained for isothermic surfaces. Surfaces of constant conformal Gaussian curvature J admit three-parameter groups of conformal motions on themselves. The Dupin cyclides are those surfaces for which the lines of curvature form a Euclidean Cartesian net relative to the conformal element of length. These are surfaces of zero curvature and are conformally equivalent to a right circular cone, a right circular cylinder or a proper torus. If $J=$constant on a gradient surface, it is isothermic. Analytic conditions are obtained for gradient (and also isothermic) surfaces along which $J=$constant. Every isothermic gradient surface is conformally equivalent to a conical surface, a cylindrical surface, or a surface of revolution.

The author also studies hypersurfaces V_n imbedded in a conformal Euclidean space R_{n+1}. *J. DeCicco.*

Davies, E. T. Subspaces of a Finsler space. Proc. London Math. Soc. (2) **49**, 19–39 (1945). [MF 13707]

This paper aims to give a simplified treatment of the subspaces of a Finsler space. As in the Riemannian case, there are, on a subspace, an intrinsic connection and an induced connection, of which the author makes a comparison. By introducing an absolute differential operator along the subspace the author generalizes the tensors of Eulerian curvature and derives formulas which correspond to the formulas of Gauss, Codazzi, and Kühne. The deformation of the subspace is studied. Several definitions are given as possible generalizations of the second fundamental form. Finally, the minimal subspaces are studied. *S. Chern.*

Laptew, B. Une forme invariante de la variation et la dérivée de S. Lie. Bull. Soc. Phys.-Math. Kazan (3) **12**, 3–8 (1940). (Russian. French summary) [MF 13902]

As the fundamental integral determining the metric of a Finsler space, the author takes the integral

$$(1)\qquad s=\int_{A_1}^{A_2} L(x^\alpha, dx^\alpha/dt)dt$$

of the calculus of variations, where L denotes a positive scalar function of the $2n$ variables x^α, dx^α/dt, $\alpha=1, \cdots, n$, positively homogeneous of dimension one with respect to the n variables dx^α/dt. In this geometry, the variation of order m of (1) is given in invariant form as the integral of the Lie derivative of order m of $L(x^\alpha, dx^\alpha/dt)$ in the vector field $v^\alpha(x)$. The actual calculation of the Lie derivative gives invariant representations of the variation of order m in terms of geometric objects and covariant derivatives; the program is carried out in detail for the second order variation. *E. F. Beckenbach* (Los Angeles, Calif.).

Clark, R. S. Projective collineations in a space of K-spreads. Proc. Cambridge Philos. Soc. **41**, 210–223 (1945). [MF 13420]

The paper extends the results of the theory of projective collineations in a space of paths to a space of K-spreads. It is shown that most of the existence theorems can be carried over without any change. The conditions for the projective equivalence of two spaces of K-spreads are also given. *M. S. Knebelman* (Pullman, Wash.).

Debever, Robert. Sur quelques problèmes de géométries dérivées du calcul des variations. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **28**, 794–808 (1942). [MF 13679]

Debever, Robert. Sur quelques problèmes de géométries dérivées du calcul des variations. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **29**, 194–203 (1943). [MF 13844]

Consider the regular variational problem or Cartan space given by the $(n-1)$-fold integral $\int f(x^i, p_i)dt^1 \cdots dt^{n-1}$,

$i=1, \cdots, n$, where $z^i = z^i(t)$,

$$p_i = (-1)^{i+1}\frac{\partial(z^1, \cdots, z^{i-1}, z^{i+1}, \cdots, z^n)}{\partial(t^1, \cdots, t^{n-1})}.$$

The passage from the $2n+1$ variables f, p_i, $f^i = \partial f/\partial p_i$, connected by the relation $f = f^i p_i$, to the variables (*) $F = f$, $P^i = ff^i$, $F_i = p_i/f$ is an involutoric ($p_i = FF_i$, $f^i = P^i/F$) contact transformation, first studied by Haar and Carathéodory. The present series of papers gives applications of (*) to Cartan geometry.

Cartan associates with every hypersurface element (z, p) the Euclidean line element $ds^2 = g_{ik}(z, p)dx^i dx^k$, where the g_{ik} are defined by the relations $gg^{ik} = \partial^2 \frac{1}{2} f^2/\partial p_i \partial p_k$, $|g_{ik}| = g^{-1}$, $g_{ik} g^{kj} = \delta_i^j$ ($g \neq 0$ because f is regular). With the notation $A^i = -f\partial\sqrt{g}/\partial p_i$, Cartan also associates with (z, p) the line element (z, λ), where $\lambda^i = f((\partial f/\partial p_i)g^{-1} - A^i g^{-1})$, which is transversal to (z, p) if and only if $A^i = 0$. This result, which is fundamental for Cartan's work, and some of its implications are proved here by means of the transformation (*). Moreover, $A^i = 0$ is also sufficient to carry over properties of parallel (geodesically equidistant) hypersurfaces from Riemann spaces to Cartan and Finsler spaces. Berwald [Compositio Math. 7, 141–176 (1939); these Rev. 1, 89] generalized parallelism to hypersurfaces with $A^i \neq 0$. By means of (*) the author obtains another such parallelism which, in a sense, is dual to Berwald's. The two parallelisms coincide if and only if $A^i = 0$. *H. Busemann.*

De Donder, Th. Mouvement d'un solide dans un espace riemannien. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 8–16 (1942). [MF 13646]

The author takes a Riemannian manifold with quadratic form $(\delta s)^2 = g_{ab}\delta x^a \delta x^b$ whose coordinates are functions of a parameter τ and finds necessary and sufficient conditions to be satisfied by the functions $x^a = x^a(\tau)$ if δs is to be independent of τ in the special case where the g_{ab} are constants. *A. Schwartz* (State College, Pa.).

De Donder, Th. Mouvement d'un solide dans un espace riemannien. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 60–66 (1942). [MF 13649]

[Cf. the preceding review.] The author considers the connection between a neighborhood of a point of a Riemannian manifold and a neighborhood of the same point in the tangent manifold at that point, assuming that the coordinates in both spaces are functions of a parameter τ. The fundamental quadratic form of the tangent manifold can have constant coefficients, and the tangent space coordinate functions $x^a = x^a(\tau)$ for which δ_s is independent of τ are then studied. *A. Schwartz* (State College, Pa.).

Coutrez, Raymond. Généralisation de la dérivée covariante spinorielle d'après la théorie de la solidification de Th. De Donder. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 457–464 (1943). [MF 13852]

[The paper referred to in the title is reviewed above.] The author considers two Riemannian manifolds with quadratic forms $\delta s^2 = g_{ab}\delta x^a \delta x^b$, $a, b = 1, \cdots, r$; $\delta\sigma^2 = \gamma_{\lambda\mu}\delta\xi^\lambda\delta\xi^\mu$, $\lambda, \mu = 1, \cdots, \rho$, whose coordinates x^a and ξ^λ are functions of a parameter τ, and studies the necessary and sufficient conditions that the functions $S_{a_1 \cdots a_p \lambda_1 \cdots \lambda_q}$ be such that $S_{a_1 \cdots a_p \lambda_1 \cdots \lambda_q}\delta x^{a_1} \cdots \delta x^{a_p}\delta\xi^{\lambda_1} \cdots \delta\xi^{\lambda_q}$ is independent of τ. In order to describe these conditions, generalizations of covariant spinor derivatives are introduced. *A. Schwartz.*

De Donder, Th., et Géhéniau, J. Sur la dérivée covariante des tenseurs généralisés. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 630–633 (1942). [MF 13672]

The authors consider the simultaneous infinitesimal transformations

$$\text{(1)} \qquad \delta x^i = X^i(x, \tau)d\tau, \qquad i = 1, \cdots, n,$$

$$\text{(2)} \qquad \delta Q_\mu = Y_\mu(x, Q, \tau)d\tau, \qquad \mu = 1, \cdots, m,$$

where X^i are arbitrary functions of the n variables x^j (space variables) and the arbitrary parameter τ (curve variable). By imposing two conditions on the Y_μ, they show that (2) may be written as $\delta Q_\mu = -A_\mu{}^\nu Q_\nu d\tau$, where $A_\mu{}^\nu$ are linear functions of the $X^j{}_{,i} = \partial X^j/\partial x^i$. Thus the infinitesimal transformation of "mixed" tensors is the sum of two transformations

$$\delta Q_{\mu \cdot i} = -(Q_{\nu \cdot i}A_\mu{}^\nu + Q_{\mu \cdot j}X^j{}_{,i})d\tau.$$

The essential problem of the paper is to determine the law of infinitesimal transformation of the "connection." That is, if the covariant derivative of Q_μ is denoted by

$$Q_{\mu \cdot i} = Q_{\mu, i} - K^\nu_{\mu i}Q_\nu,$$

then how do the $K^\nu_{\mu i}$ transform? This question is completely answered. *N. Coburn* (Austin, Tex.).

Samuel, Pierre. Sur les tenseurs à dérivées covariantes nulles. C. R. Acad. Sci. Paris 220, 160–162 (1945). [MF 13491]

Dans un espace de Riemann les dérivées covariantes du tenseur métrique $g_{\alpha\beta}$ sont nulles. L'auteur cherche s'il existe d'autres tenseurs à dérivées covariantes nulles. Il se borne à des tenseurs de la forme $b_\beta{}^\alpha$ et suppose que ce tenseur est réductible à la forme diagonale. Il obtient les résultats suivants. Si $b_{\alpha\beta} = g_{\alpha\gamma}b_\beta{}^\gamma$ est symmétrique, il est possible de décomposer le ds^2 en une somme de k éléments linéaires indépendants, où k est le nombre des valeurs propres de $b_\alpha{}^\beta$. Dans le cas général il existe un répère holonome formé de directions propres de $b_\beta{}^\alpha$ avec les propriétés suivantes: $g_{\lambda\mu}$ ne depend que des u^λ, $g_{\lambda a}$ est constant. Ici $\lambda, \mu, \cdots$ et $a, b, \cdots$ sont des indices relatifs à les deux valeurs propres réelles $\underset{\lambda}{s}$ et $\underset{a}{s}$ ($\underset{\lambda}{s} \neq \underset{a}{s}$). Les g_{ab}, qui peuvent être différents de zéro, sont indiqués, de même que les $g_{\lambda\mu}$ d'un sous-espace déterminé par deux valeurs propres conjugées. *J. Haantjes* (Amsterdam).

Wade, T. L. Tensor algebra and invariants. II. Nat. Math. Mag. 20, 5–10 (1945). [MF 13978]

This is a continuation of a previous paper by the author [same Mag. 19, 3–10 (1944); these Rev. 6, 107]. In the first section of this paper, the author reviews the work of previous authors on numerical tensors (that is, tensors constructed from the Kronecker delta). These include the determinant of the deltas, the permanent tensor, immanent tensors of the symmetric group, and the ϵ systems. The next section is devoted to a brief discussion of Cramlet's theory for constructing projective concomitants. The theory is illustrated by several examples. In the final section, the author discusses various geometries which are subgeometries of projective geometry as theories of an adjoined numerical tensor. This point of view is illustrated by considering affine geometry, Euclidean (parabolic metric) geometry, and non-Euclidean (elliptic metric) geometry. A typical theorem is the following: "elliptic metric geometry as a subgeometry of projective geometry is the theory of the tensor ϵ^{ij}." For more complete discussions of these topics, references to the literature are furnished. *N. Coburn* (Austin, Tex.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Brown, F. Gilbert. Exact addition formulae for the axial spherical aberration and curvature of field of an optical system of centred spherical surfaces. Proc. Phys. Soc. 57, 403–411 (1945). [MF 13980]

The formulae show the contribution of each surface to the aberration concerned, in a form intended to be useful in controlling the aberration during design. Expressions are presented for exact spherical aberration of a ray of finite aperture, ratio of the magnification of the marginal to that of the paraxial ray, Petzval curvature of field for central rays of finite field angle, and sagittal astigmatism. The Petzval surface considered "exact" is obtained by imaging the finite object surface by thin bundles through the center of curvature of the first lens surface, re-imaging the result by thin bundles through the center of curvature of the second surface, and so on. [There are several misprints or mistakes in notation, an inconsistency in sign convention for spherical aberration, and an error of signs in the series expansion of the arcsin function and in a formula depending on it.] The author mentions that he has prepared tables of $\sin I-\sin I'$ as a function of the deviation $I-I'$ and the relative refractive index n'/n.

A. J. Kavanagh (Buffalo, N. Y.).

Bateman, H., and Pekeris, C. L. Transmission of light from a point source in a medium bounded by diffusely reflecting parallel plane surfaces. J. Opt. Soc. Amer. 35, 651–657 (1945). [MF 13775]

A point source of light is placed in a medium between two plane parallel plates which are diffusely reflecting. The total luminous flux incident upon a spherical receiver in the same medium is to be found. This involves the solution of a pair of simultaneous integral equations for the illuminances E and E' of the plane surfaces. The authors solve these equations exactly by means of a Fourier transformation of E and E'. From this result the flux upon the receiver is obtained and represented by an integral involving Bessel functions. When the distance of the receiver from the point source is large the integral is transformed into a form suitable for numerical evaluation, and an asymptotic expression is derived.

R. K. Luneberg (Buffalo, N. Y.).

de Mallemann, R., et Suhner, F. Réflexion elliptique normale et oblique, étude optique des couches minces. Rev. Optique 23, 20–38 (1944). [MF 13968]

The reflection of light by a plane surface which separates two isotropic or anisotropic media is investigated. Formulae for the reflectance are given in the case of an isotropic medium against a uni-axial crystal. The fact that the results predicted by these formulae with regard to the polarization of the reflected light are not thoroughly verified by careful measurements is generally explained by the assumption of a transition layer between the two media. In view of this the authors consider the problem of a thin homogeneous layer (anisotropic in general) between the two media and derive formulae for the reflectance of such a layer. These formulae can be considered as a good first approximation to the actual situation. A measuring instrument for the verification of their results is described in the last part of the paper.

R. K. Luneberg (Buffalo, N. Y.).

Robin, Louis. Sur un problème de diffraction d'ondes électromagnétiques à la surface de séparation de deux milieux. C. R. Acad. Sci. Paris 218, 135–136 (1944). [MF 13461]

The problem of diffraction at the plane interface between two nonconducting media was discussed by Delsarte [same C. R. **202**, 826–828, 1026–1028 (1936); Ann. Sci. École Norm. Sup. (3) **53**, 223–273 (1936)], the solutions not being restricted to sinusoidal functions of the time. He considered this problem both for the scalar equation of wave motions and for Maxwell's electromagnetic equations. The present note deals with the same problem for the scalar equation of wave motions when the media are conducting. The note is extremely abbreviated and details are promised in a paper to be published in another journal.

E. T. Copson (Dundee).

Duffieux, Michel, Tirouflet, Jean, Guenoche, Henri, et Lansraux, Guy. Image d'une fente en éclairage coherent. Ann. Physique (11) 19, 355–395 (1944). [MF 13715]

This paper is concerned with the following problem. When plane waves of monochromatic light are incident on a slit, they produce a Fraunhofer diffraction pattern. It is required to find the effect when this pattern is incident on a second parallel slit. The investigation is both theoretical and experimental, but its interest is mainly experimental. The theory, as the authors state, is essentially that given by Rayleigh [Philos. Mag. (6) **33**, 161–178 (1917)]. The authors compute the effect on the basis of Rayleigh's formulae and show that the results are in agreement with their experiments.

E. T. Copson (Dundee).

Corso Lopez de Romaña, José Maria. On the delay in the passage of waves of light in passing through a transparent plate. Revista Univ. Católica Perú 11, 138–140 (1943). (Spanish) [MF 14185]

Chandrasekhar, S. The formation of absorption lines in a moving atmosphere. Rev. Modern Phys. 17, 138–156 (1945). [MF 13687]

Schuster's problem in a moving stellar atmosphere is considered for the case in which the mass scattering coefficient $\sigma(\nu)$ is constant for $\nu_0-\Delta\nu\leqq\nu\leqq\nu_0+\Delta\nu$ and is zero otherwise and in which the velocity normal to the planes of stratification of the atmosphere increases linearly with the optical depth. The first approximation of the equation of transfer is discussed by the method of replacing integrals by sums according to Gauss's formula for numerical quadratures, already developed by the author. The solution of the problem depends on that of a differential equation of hyperbolic type with an unusual boundary condition. Certain illustrative cases of the line contours of the absorption lines formed in such a moving atmosphere are worked out numerically.

G. C. McVittie (London).

Hammad, A. The passage of sunlight through the earth's atmosphere. General analysis and formulation of the problem. Proc. Math. Phys. Soc. Egypt 1, no. 4, 1–9 (1940). [MF 14106]

This paper deals with a problem which was much more completely treated in the classical paper of L. V. King [Philos. Trans. Roy. Soc. London. Ser. A. **212**, 375–433 (1913)]. There seem to be no new results.

W. E. K. Middleton (Toronto, Ont.).

Bloch, Léon. Remarques sur la nouvelle théorie de la lumière. J. Phys. Radium (8) 6, 196–202 (1945). [MF 13791]

In De Broglie's theory of light [Nouvelle Théorie de la Lumière, Paris, 1940] a photon consists of two particles, called semi-photons, one of which obeys Dirac's equation and the other a similar equation. The author points out that the two equations can be transformed into one another by an improper Lorentz transformation, for example, the one which transforms the world-point (x, y, z, t) into $(-x, y, -z, -t)$. Only if the semi-photons occupy (on the average) positions related to each other by such a transformation does a photon exist. This new interpretation changes nothing essential in De Broglie's theory.

P. Weiss (London).

Bloch, Léon. Remarques sur la nouvelle théorie de la lumière. C. R. Acad. Sci. Paris 220, 109–111 (1945). [MF 13488]

De Broglie has proposed a theory of light in which the photon is regarded as the result of the fusion of two "demi-photons," one of which satisfies Dirac's equation and the other the complementary equation. The result is that the demi-photons must be regarded as having masses equal but of opposite sign. The author points out that the same equations are obtained if axes of space and time with opposite signs are used in one equation as compared with the other. The case when the two equations represent the same particle is considered and shown to lead to the Maxwellian theory of the electromagnetic field. *G. C. McVittie.*

Bloch, Léon. Sur une identité de la théorie du photon. C. R. Acad. Sci. Paris 220, 240–241 (1945). [MF 13499]

The equation $\sigma_4 f_4 - \sigma_x f_x - \sigma_y f_y - \sigma_z f_z = 0$ has been verified by De Broglie in the case of the electron for a plane monochromatic wave. It is now shown that, in the case of the photon, f_x, σ_x, etc. must be replaced by $\frac{1}{2}(f_x{}^A B_4 + f_x{}^B A_4)$, $\frac{1}{2}(\sigma_x{}^A B_4 + \sigma_x{}^B A_4)$, etc., where A and B denote that De Broglie matrices must replace the Dirac matrices.

G. C. McVittie (London).

Ma, S. T. Fourier transforms of retarded and advanced potentials. Phys. Rev. (2) 68, 166–172 (1945). [MF 13789]

Using the well-known Jordan-Pauli invariant delta-function [Z. Physik 47, 151–173 (1928)] and Dirac-Pauli D-functions [Pauli, Phys. Rev. (2) 58, 716–722 (1940)], the author derives the expressions for the kernel $g(X, X')$ of the Fourier transforms of the advanced and retarded potentials for electromagnetic and particle-wave fields. It is shown that Cauchy's principal value of the integral for the kernel agrees almost everywhere (except at $r_0 = 0$) with $g(X, X')$ obtained directly from the defining integrals for the potentials. The relation of the transforms to Dirac's inward- and outward-moving waves is discussed. *C. Kikuchi.*

Frenkel, J. On the theory of seismic and seismoelectric phenomena in a moist soil. Acad. Sci. USSR. J. Phys. 8, 230–241 (1944). [MF 14162]

Translation of a paper in Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 8, 133–150 (1944); these Rev. 6, 223.

Grivet, Pierre. Nouvelle méthode pour calculer les propriétés des résonateurs électromagnétiques. C. R. Acad. Sci. Paris 218, 71–73 (1944). [MF 13452]

The frequency ω of the lowest characteristic oscillation of a cavity resonator can be determined by methods of the calculus of variations. In fact, if E and H are the space factors of an electric or magnetic field satisfying appropriate boundary conditions, the minimum of either of

$$\int_V (\operatorname{curl} E)^2 d\tau \bigg/ \int_V E^2 d\tau, \qquad \int_V (\operatorname{curl} H)^2 d\tau \bigg/ \int_V H^2 d\tau$$

is ω^2/c^2. An estimate (upper bound) for ω can be obtained by Rayleigh's method. If the field distribution is also needed, the Ritz method can be applied to minimise either of the two expressions. From the above formulae the variation $\delta\omega$ of the frequency due to a variation δV of the cavity can also be obtained in the form

$$\delta\omega = \tfrac{1}{2}\omega \int_{\delta V} (E^2 - H^2)\, d\tau \bigg/ \int_V H^2 d\tau.$$

A. Erdélyi (Edinburgh).

Grivet, Pierre. Sur la longueur d'onde propre de certains résonateurs électromagnétiques. C. R. Acad. Sci. Paris 218, 183–185 (1944). [MF 13477]

The method indicated in the preceding review is applied to a number of simple shapes such as the cylinder, the "ring" of rectangular cross section, the "rhumbatron" (space between two cylinders of finite length, one inside the other, with common axis and common base) and the sphere. In the first two cases both the Rayleigh and the Ritz methods (the latter with two adjustable parameters) are applied, in the last two cases only the Rayleigh approximation. In the case of an infinite cylinder, if b is the ratio of the wave length of the lowest oscillation to the radius, the Rayleigh approximation $E_z = \cos(\pi r/2a)$ (r, the distance from the axis) gives the approximation $b = 2.602$; the Ritz approximation $E_z = \frac{1}{2}(A + Br)(r^2 - a^2)$ with two parameters A, B gives the approximation $b = 2.6105$; the correct value is $b = 2.61$. *A. Erdélyi* (Edinburgh).

Warnecke, Robert, et Bernier, Jean. Génération électronique d'ondes électromagnétiques dans un résonateur creux. C. R. Acad. Sci. Paris 218, 73–75 (1944). [MF 13453]

The authors discuss the forced oscillations generated by an electronic current ρv in a cavity resonator. The electromagnetic field can be represented by a vector potential and a scalar potential. If the vector potential can be represented by a series of "normal solutions" $\sum_\nu q_\nu(t) a_\nu(P)$, where t is time, P the point of observation and $a_\nu(P)$ a suitable solution of the wave equation $\Delta a + (\omega/c)^2 a = 0$, then for the time factor the authors obtain from the Maxwell-Lorenz equations the ordinary differential equation

$$\ddot{q}_\nu + (\omega_\nu/S_\nu)\dot{q}_\nu + \omega_\nu{}^2 q_\nu = 4\pi c \int \rho v a_\nu d\tau,$$

the integration being extended over the volume occupied by the electrons. *A. Erdélyi* (Edinburgh).

Bernier, Jean. Une méthode de raccordement pour le calcul des cavités électromagnétiques. C. R. Acad. Sci. Paris 218, 186–188 (1944). [MF 13466]

Consider a cavity consisting of two adjoining parts V_1 and V_2 which have a surface Σ in common; S_1 and S_2 are

the metallic boundaries of V_1 and V_2. Suppose that the general solution of the wave equation is known for V_1 and V_2 and let the solutions satisfying the boundary conditions appropriate to the vector potential be A_1 and A_2, with tangential components A_{1t} and A_{2t} on S_1 and S_2. The actual field in the combined cavity V_1+V_2 minimises the expression

$$J=\int_\Sigma (A_1-A_2)^2 d\sigma+\int\int_{S_1} A_{1t}^2 d\sigma+\int\int_{S_2} A_{2t}^2 d\sigma,$$

the actual minimum being zero. If A_1 and A_2 are represented by infinite series with unknown coefficients, J is a quadratic form in these coefficients and the minimum problem leads to an infinite system of linear equations. The author has calculated by his method the fundamental oscillation (of electric type) of a U-shaped cavity.

A. Erdélyi (Edinburgh).

Müller-Strobel, Josef, und Patry, J. Berechnung von Hilfsfunktionen für gerade Empfangsantennen beliebiger Höhe. Helvetica Phys. Acta 17, 455–462 (1944).

In a previous paper [same Acta 17, 127–132 (1944); these Rev. 6, 55] the authors derived approximate formulas for the input current of straight aerials, based on Hallén's theory. In the present paper, graphs and numerical approximations are given for some auxiliary functions which occur in those expressions. These auxiliary functions are in terms of elementary transcendentals and of the sine and cosine integrals of the arguments L and $2L$, where $L=2\pi l/\lambda$ (l, the length of the aerial; λ, the wave length). While in the previous paper the constant, linear and quadratic terms of the field distribution along the aerial were taken into account, each term involving two auxiliary functions, only those connected with the constant term are discussed here. As previously, copper losses and end-capacities are neglected as well as higher powers than the first of $\Omega=2\log(2l/\rho)$.

H. G. Baerwald (Cleveland Heights, Ohio).

Lüdi, F. Zur Theorie der Richtstrahlung mit Parabolspiegeln. Helvetica Phys. Acta 17, 374–388 (1944).

The problem considered is the determination of the radiation field of a dipole at the focus of a parabolic mirror, the focus being in the plane of the mirror aperture. Since the spherical field of the dipole is reflected from the mirror as an equiphase plane wave the problem reduces to that of the diffraction of a plane wave at a circular aperture. Kirchhoff's integral formula is used to determine the distant field, some approximations being introduced to simplify the integrals and express the field in closed form. Radiation patterns in the two principal planes are drawn and shown to compare favorably with experimental results. The directive gain of the dipole-mirror system is given by the formula $2.65R/\lambda$, where R is the radius of the aperture. The author discusses the similarity of his results to those obtained for sectoral horns by Barrow and Lewis [Proc. I.R.E. 27, 41–50 (1939)] and for saw-tooth antenna arrays by Chireix [L'Onde Électrique 5, 237–262 (1926)].

M. C. Gray.

Borgnis, F. Zur Elektrostatik des Elektronenstrahls von kreisförmigem Querschnitt. Ann. Physik (5) 43, 616–629 (1943). [MF 13388]

The author calculates the potential and electric field produced by a space charge cloud in the region between two parallel infinite perfectly conducting planes at the same potential. The charge density ρ is confined to a circular cylinder of radius R with its axis perpendicular to the plates and is a function of x alone, where x is the coordinate along this axis. Thus if ρ is expressed by the Fourier series $\rho=(\epsilon/4\pi)\sum_n b_n \sin n\pi x/d$, where d is the distance between the plates, the potential ϕ is found, by solution of Poisson's equation, to be

$$\phi=(d/\pi)^2\sum_n n^{-2}b_n\{1+(n\pi^2R/2d)\times H_1^{(1)}(in\pi R/d)J_0(in\pi r/d)\}\sin n\pi x/d,\quad r\leqq R;$$

$$\phi=(d/\pi)^2\sum_n n^{-2}b_n\{(n\pi^2R/2d)\times J_1(in\pi R/d)H_0^{(1)}(in\pi r/d)\}\sin n\pi x/d,\quad r\geqq R.$$

By going to the limit of a point charge, what amounts to the Green's function G for the region is calculated. It is well known, however, that G can be determined directly for such a region and hence the potential for arbitrary charge density can be obtained by Green's theorem. The special case considered by the author follows at once.

The author also extends his calculations to a cylindrical space charge cloud along the axis of a metallic circular cylinder. Again the result was obtained directly from Poisson's equation; the reviewer feels that the use of Green's function appropriate to this region would have led to more general results without extra labor.

D. S. Saxon.

Davy, N. Ten two-dimensional electrostatic problems. Philos. Mag. (7) 36, 153–169 (1945). [MF 13338]

The author applies the Schwarz-Cristoffel transformation to the solution of various two-dimensional electrostatic problems involving rectangular conductors and finite or semi-infinite thin plates at assigned potentials. In most cases only the results are given, these including the field strengths, surface densities and total charges, capacities between different parts of the system and forces acting on small magnetic bodies at any point of the field. Diagrams of experimentally obtained equipotentials are included. A typical problem solved by this method is that of a semi-infinite rectangular conductor with equal finite thin projecting plates. In a final section the author gives a tentative solution of the problem of two finite unequal thin collinear plates at potentials $\pm V_0$.

M. C. Gray.

Parodi, Maurice. Sur la relation de Carson. Revue Sci. (Rev. Rose Illus.) 81, 26–27 (1943). [MF 13810]

The author discusses the use of the Faltung theorem in the calculation of transient currents and voltages in a transmission line.

A. E. Heins (Cambridge, Mass.).

Parodi, Maurice. Propagation sur un câble comportant seulement de la résistance et de la capacité, ces paramètres étant fonctions de l'espace et satisfaisant à certaines relations. C. R. Acad. Sci. Paris 221, 257–259 (1945). [MF 14494]

Raymond, François. Sur un théorème de la théorie des réseaux polyphasés. C. R. Acad. Sci. Paris 218, 148–150 (1944). [MF 13465]

A theorem of Julia and Fallou [same C. R. 202, 1767–1769 (1936)] is shown to be a special case of a general theorem.

A. E. Heins (Cambridge, Mass.).

Quantum Mechanics

Dirac, P. A. M. On the analogy between classical and quantum mechanics. Rev. Modern Phys. **17**, 195–199 (1945). [MF 13690]

A method is given of defining general functions of two or more noncommuting observables in the author's form of quantum mechanics. If $f(a, b, \cdots)$ is a complex-valued function of the real variables $a, b, \cdots$, it is shown that these real variables can be replaced one at a time by observables $\alpha, \beta, \cdots$. There results an operator-valued function $f(\alpha, \beta, \cdots)$ of observables. The function obtained depends, in general, on the order in which the replacements are made, unless the observables commute. It is then possible to define a formal "probability" that observables (such as the coordinates) of a dynamical system in a given quantum state have given numerical values or values that lie within given ranges. This probability is in general a complex number, but it may be given some meaning, since when it is close to zero one can say that the numerical values are unlikely. It also depends on an order among the observables if they do not commute, but this may be taken as the temporal order if they refer to different times. One may then define the probabilities of different trajectories for the motion of a particle in quantum mechanics, which makes for a closer resemblance between quantum mechanics and classical mechanics. The author also uses this method to set up in a more general fashion the analogy between classical and quantum-mechanical contact transformations.

O. Frink (State College, Pa.).

Born, Max. On the quantum theory of pyroelectricity. Rev. Modern Phys. **17**, 245–251 (1945). [MF 13692]

Use is made of the method of solving the wave equation for any atomic system developed by Born and Oppenheimer in 1927. An arbitrary configuration of the nuclei (coordinates X) is assumed and the wave equation is solved for the electrons (coordinates x). The energy in every electronic state is then a function of the nuclear coordinates X and is, say, $\Phi(X)$ for the lowest electronic level. The solution of the actual wave equation in which the x and X both are variables is then expanded in powers of the parameter $\tau=(m/M)^{\frac{1}{4}}$, where m is the mass of the electron, M that of a nucleus. For $\tau=0$ the eigenfunction can be written $\psi(x, X)=\chi(X)\phi(x, X)$. The condition for the solvability of the next approximation is that $\Phi(X)$ is stationary. With the wave function $\phi(x, X)$ for the electronic ground state the intermediate diagonal element is

$$M_a(X)=\int \phi^*(x, X)M_a\phi(x, X)dx,$$

where M_a is the total electric moment.

The older theories leading to a T^4 law for the dependence of electric moment on the temperature T are first sketched and this result, which is not in agreement with experiment, is associated with the occurrence of terms involving integrals of the type

$$\int_0^{w_f} \frac{\hbar w^3 dw}{e^{\hbar w/kT}-1}.$$

Also the hypothesis of rigid ions is a very poor approximation even in the case of ionic crystals. Actually $M_a(X)$ is not simply a linear function of the X. With displacements u the quantity $M_a(X^0+u)$ can be expanded in a power series of the u and in a crystal $u=u_{\text{hom}}+u_{\text{therm}}$, thermal vibrations being superposed on a homogeneous deformation. While it is sufficient to take only linear terms in u_{hom}, higher terms of u_{therm} must be considered. The u_{therm} can be replaced by the normal coordinates ξ and when M_a is expanded there is a term $M_a(\xi)$ in addition to those involving the strain components; this $M_a(\xi)$ may be regarded as giving the "true" pyroelectricity. It is shown to give the correct dependence on T, namely a T^2 law. Indeed, this is found to depend on the occurrence of terms of type

$$\int_0^{w_f} \frac{\hbar w dw}{e^{\hbar w/kT}-1}$$

(w_f=Debye's maximum frequency), for which Debye's table of an associated integral can be used.

H. Bateman.

Humblet, J. L'énergie d'interaction et la théorie du rayonnement multipolaire. Physica **11**, 91–99 (1944). (French. English summary)

Given a system of charged particles in an electromagnetic field, the author obtains a complete expansion giving the interaction energy of a transition between quantum states corresponding to any 2^n-pole electric or magnetic moment. He then uses this expansion as the basis for a quantum theory of the emission and absorption of radiation by "multipolar processes." The method used is to develop in Taylor's series the electric and magnetic potentials, as well as the elements of the Dirac interaction-energy matrix. In the special cases of the dipole and quadrupole moments, the formulas obtained are the same as those previously found by Møller and Rosenfeld, who used a similar method. The expansions for the 2^n-pole moments involve nth order partial derivatives of the potentials.

O. Frink.

Humblet, J. Théorie de l'émission et de l'absorption multipolaires. Physica **11**, 100–113 (1944). (French. English summary)

Using the results of the paper reviewed above, giving the interaction energy of a transition between states corresponding to the general electric or magnetic 2^n-pole moment, the author derives transition probabilities for the emission and absorption of radiation by such a multipolar process. He shows that the probabilities for an electric 2^{n+1}-pole process are of the same order of magnitude as those for a magnetic 2^n-pole process, and that there is no interference between two such processes. The mathematics used involves spherical harmonics and the associated Legendre polynomials. It is shown that, for the special case of emission and absorption by hydrogen-like atoms, the formulas may be considerably simplified.

O. Frink.

Gustafson, Torsten. On the elimination of divergencies in quantum field theory. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **15**, no. 28, 277–288 (1945). [MF 14318]

This is an application of the quantum field theory of positive and negative energy photons, originally proposed by Dirac [Proc. Roy. Soc. London Ser. A. **180**, 1–40 (1942)] and later discussed in detail by Pauli [Rev. Modern Phys. **15**, 175–207 (1943); these Rev. **5**, 277]. As an example of the method, the integrals that arise in the calculation of the Compton effect are discussed. The author gives the formal expression for the transition probability to the fourth-order approximation and shows that the second-order terms, which diverge in the usual formulation, are convergent. The fourth-order terms are to be discussed in a later paper.

C. Kikuchi (East Lansing, Mich.).

Rosenfeld, Léon. Sur la définition du spin d'un champ de rayonnement. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 562–568 (1942). [MF 13670]

This article is concerned with the removal of an ambiguity in the classical theory of the spin. In any field theory which is relativistically invariant and satisfies a variation principle a symmetrical energy-momentum tensor T^{ik} may be obtained by varying the Lagrangian with respect to the metric, and another nonsymmetrical tensor $\bar{T}^{ik}$ may be obtained by applying the principles of the calculus of variations. Both tensors have a vanishing divergence as required by the conservation laws, and they differ from each other by a tensor which is the divergence of another tensor $R^{ij,k}$ which is skew-symmetrical in the indices i and j. This tensor gives rise to the following two alternative definitions of the spin tensor:

$$M_s^{ik}=\int\left(x^k\frac{\partial R^{ij,i}}{\partial x^j}-x^i\frac{\partial R^{ij,k}}{\partial x^j}\right)dv,\ \bar{M}_s^{ik}=\int(R^{4i,k}-R^{4k,i})dv,$$

$i, k=1, 2, 3$, the integration being extended over a 3-dimensional space-like section. It is shown that both definitions lead to the same conclusion in the two physically important cases when either the field vanishes at infinity to a sufficiently high order or it is periodic in time and temporal mean-values are taken. *P. Weiss* (London).

Beck, Guido. Field concepts in quantum theory. Rev. Modern Phys. 17, 187–194 (1945). [MF 13689]

After discussing the present state of quantum electrodynamics by means of dimensional considerations, the author puts forward a mathematical scheme based on the 16 Dirac-Eddington matrices which is to replace Maxwell's electrodynamics of the vacuum. States are described by four-fold matrices ψ with conjugates $\bar{\psi}$, and kinematical quantities by $\{M\}=\bar{\psi}M\psi$, where M denotes any one of the 16 Dirac-Eddington matrices. Besides the usual differentiation

$$\left\{\frac{\partial M}{\partial \xi}\right\}=\frac{\partial\bar{\psi}M}{\partial\xi}\psi+\bar{\psi}\frac{\partial M\psi}{\partial\xi},$$

the author introduces a second type of differentiation,

$$\left\{\frac{\delta M}{\delta \xi}\right\}=\frac{\partial\bar{\psi}M}{\partial\xi}\psi-\bar{\psi}\frac{\partial M\psi}{\partial\xi}.$$

A set of 10 differential equations, involving both types of differentiation, is then obtained and an attempt to interpret them is made. *P. Weiss* (London).

Beck, Guido. Polarisation of the vacuum by a discontinuous potential. Revista Unión Mat. Argentina 11, 18–29 (1945). (Spanish. English summary) [MF 14099]

The considerations recently given on the microkinematics implied by Dirac's theory [see the preceding review] have, so far, only been applied to the case of the vacuum. The present paper shows how the influence of an external electromagnetic field on the state of the vacuum can be investigated and considers in detail a simplified, one-dimensional model of a discontinuous electrostatic potential. *Author's summary.*

Bertaut, Félix. Sur la validité du théorème d'Ehrenfest en mécanique ondulatoire de Dirac. C. R. Acad. Sci. Paris 220, 105–107 (1945). [MF 13486]

Ehrenfest's theorem states that Newton's second law of motion still holds in nonrelativistic wave-mechanics in matrix form. The author attempts to extend this theorem to Dirac's relativistic theory. His result does not contain the spin of the electron. It states that the rate of change of the electron's proper momentum equals the space-integral of the Lorentz force. *P. Weiss* (London).

Stueckelberg, E. C. G. La charge gravifique et le spin de l'électron classique. Helvetica Phys. Acta 18, 21–44 (1945).

The present paper is an extension of the author's earlier works [same Acta 14, 51–80 (1941); 17, 3–26 (1944)] on the theory of the electron to nonlinear fields. The principal results are (1) that the gravitational charge of the whole system is equivalent to its inertial mass and (2) that the equations for the world-line of the electron possess periodic solutions even in the absence of incident radiation. It is interesting to note that in this theory accelerated charges do not necessarily radiate. *C. Kikuchi.*

Stueckelberg, Ernest-C.-G. Solutions invariantes $D_s(x, y)$ de l'equation $(\square-\kappa^2)D=0$ dans l'espace pseudo-euclidien. C. R. Séances Soc. Phys. Hist. Nat. Genève 59, 49–52 (1942). [MF 14198]

Stueckelberg, Ernest-C.-G. Solutions invariantes $D_s(x, y)$ de l'équation de Schroedinger relativiste. C. R. Séances Soc. Phys. Hist. Nat. Genève 59, 53–55 (1942). [MF 14199]

Stueckelberg, Ernest-C.-G. Principe de correspondance d'une mécanique asymptotique quantifiée. C. R. Séances Soc. Phys. Hist. Nat. Genève 61, 159–161 (1944). [MF 14224]

Stueckelberg, Ernest-C.-G., et Bouvier, Paul. Le freinage de radiation de l'électron de Dirac en mécanique asymptotique. C. R. Séances Soc. Phys. Hist. Nat. Genève 61, 162–165 (1944). [MF 14225]

Arnous, Edmond. La fonction caractéristique quantique et la méthode des perturbations. C. R. Acad. Sci. Paris 220, 348–349 (1945). [MF 13947]

Arnous, Edmond. La fonction caractéristique quantique et les méthodes d'approximation du genre champ self consistent. C. R. Acad. Sci. Paris 220, 440–442 (1945). [MF 13959]

de Beauregard, Olivier Costa. Sur la théorie des grandeurs non simultanément mesurables. C. R. Acad. Sci. Paris 221, 256–257 (1945). [MF 14493]

Courtel, Robert. Sur la perturbation d'un problème de valeurs propres par modification de la frontière; cas des équations de la mécanique ondulatoire. C. R. Acad. Sci. Paris 221, 346–347 (1945). [MF 14505]

Durand, Émile. Sur les identités quadratiques de la théorie de Dirac. C. R. Acad. Sci. Paris 220, 517–520 (1945). [MF 14051]

Kwal, Bernard. La mécanique multi-ondulatoire. C. R. Acad. Sci. Paris 220, 844–846 (1945). [MF 14070]

Kwal, Bernard. Le principe fondamental de la mécanique ondulatoire relativiste et la théorie des corpuscules limites. C. R. Acad. Sci. Paris 220, 905–907 (1945). [MF 14074]

Kwal, Bernard. Le principe fondamental de la mécanique ondulatoire relativiste et les équations d'onde associées au moment de la quantité de mouvement. C. R. Acad. Sci. Paris 221, 95–97 (1945). [MF 14243]

Morette, Cécile. Sur les ensembles de fonctions d'ondes possibles correspondant à des conditions initiales mal déterminées. C. R. Acad. Sci. Paris 220, 487–488 (1945). [MF 14045]

Slansky, Serge. Sur quelques points du problème des deux corps en mécanique ondulatoire relativiste. C. R. Acad. Sci. Paris 220, 491–493 (1945). [MF 14047]

Slansky, Serge. La généralisation des transformations de Lorentz et les équations d'ondes d'un système. C. R. Acad. Sci. Paris 220, 551–553 (1945). [MF 14054]

Viard, Jeannine. Sur les intégrales premières en mécanique ondulatoire. C. R. Acad. Sci. Paris 221, 93–95 (1945). [MF 14242]

Viard, Jeannine. Sur la théorie relativiste du nucléon et l'interprétation du spin isotopique. C. R. Acad. Sci. Paris 220, 300–302 (1945). [MF 13945]

Viard, Jeannine. Expression générale des opérateurs fondamentaux attachés à un système formé de deux corpuscules de Dirac. C. R. Acad. Sci. Paris 220, 494–496 (1945). [MF 14048]

Mariani, M. Géométrie métrique et corpuscules élémentaires. II. J. Phys. Radium (8) 1, 322–334 (1940). [MF 13984]

[Part I appeared in the same J. (7) 10, 296–306 (1939).] It is assumed that in the interior of an atomic nucleus, $r<a$, where $a\sim 10^{-13}$ cm., the line-element is that of de Sitter,

$$ds^2=c^2(1-r^2/a^2)dt^2-r^2(\sin^2\vartheta d\varphi^2+d\vartheta^2)-(1-r^2/a^2)^{-1}dr^2;$$

while in the exterior, $r>a$, it is given by

$$ds^2=c^2(1-a^2/r^2)dt^2-r^2(\sin^2\vartheta d\varphi^2+d\vartheta^2)-(1-a^2/r^2)^{-1}dr^2.$$

The properties of the spherical and of the elliptic space are discussed with reference to the line-element for $r<a$. The indeterminateness of the distance by an integral multiple of $2\pi a$ or πa is considered as a geometrical analogy to Bohr's quantum rule. *P. Weiss* (London).

Pauli, W., and Hu, N. On the strong coupling case for spin-dependent interactions in scalar- and vector-pair theories. Rev. Modern Phys. 17, 267–286 (1945). [MF 13695]

Two different forms of the interaction between mesons and nucleons have been proposed: linear interaction, analogous to electromagnetic interaction, and pair theory, which assumes a bilinear form of interaction. The spin dependent interaction in pair theory could previously only be treated by the perturbation method (weak coupling) and the results thus obtained are not in agreement with experiment. The strong coupling case for the spin dependent interaction is investigated in the present paper. The result proves to be unsatisfactory. It is found that the force between nucleons resulting from this theory is nearly spin independent, provided the cutting off radius is assumed to be small as compared with the separation of the nucleons. *L. Janossy* (Manchester).

Heitler, W., and Walsh, P. Theory of cosmic-ray mesons. Rev. Modern Phys. 17, 252–262 (1945). [MF 13693]

In a previous paper, Hamilton, Heitler and Peng [Phys. Rev. (2) 64, 78–94 (1943)] have discussed cosmic ray phenomena in terms of the production of mesons in collisions between nucleons. It was shown by Heitler [Proc. Roy. Irish Acad. Sect. A. 50, 155–165 (1945); these Rev. 7, 102] that the cross-sections previously used have to be modified. The collision between fast nucleons was treated by the method of Williams and Weizsäcker. In the older treatment only the mesons emitted by the stationary nucleon due to the impact of the fast nucleon were considered. In the more recent treatment it was shown that the mesons emitted by the fast nucleon caused by the perturbation due to the stationary nucleon must also be considered.

In the present paper it is shown that the main features of cosmic ray mesons can be accounted for when using the modified cross-sections. In particular, the following phenomena are treated: energy spectrum, total intensity of mesons, penetrating showers. *L. Janossy* (Manchester).

Fierz, Markus. Zur Spin-Bahnkoppelung zweier Nukleonen in der Mesontheorie. Helvetica Phys. Acta 18, 158–166 (1945).

Bleuler, Konrad. Ein Beitrag zum Zwei-Nukleon-Problem. Helvetica Phys. Acta 18, 317–342 (1945).

Wentzel, G. Anisotropie des Proton-Neutron-Streuung und symmetrische Mesontheorie. Helvetica Phys. Acta 18, 430–446 (1945).

Houriet, A. Structure du nucléon d'après les théories mésoniques à couplage serré. Helvetica Phys. Acta 18, 473–496 (1945).

Ginsburg, V., and Smorodinsky, J. On the wave equations for particles with variable spin. Acad. Sci. USSR. J. Phys. 8, 52–53 (1944). [MF 14164]

Translation of a paper in Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 13, 274–276 (1943); these Rev. 6, 111.

Ginsburg, V. L. On the theory of excited spin states of elementary particles. Acad. Sci. USSR. J. Phys. 8, 33–51 (1944). [MF 14163]

Tonnelat, Marie-Antoinette. Sur une représentation à 5 dimensions des équations des particules de spin 1 et 2. C. R. Acad. Sci. Paris 221, 173–175 (1945). [MF 14248]

Petiau, Gérard. Sur les états de masse des corpuscules de spin quelconque. C. R. Acad. Sci. Paris 220, 489–491 (1945). [MF 14046]

Géhéniau, J. Théorie variationnelle des spineurs. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 26, 44–52 (1940). [MF 13827]

Géhéniau, J. Théorie variationnelle des spineurs. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 26, 133–143 (1940). [MF 13830]

Expository articles. In the second, the author follows closely work of Th. De Donder [same Bull. (5) 10, 188–201 (1924)], and finds identities which functions of spinors must satisfy if they are to be density factors in the restrained sense. *A. Schwartz* (State College, Pa.).

Géhéniau, J. Les identités fondamentales de la physique mathématique étendues aux variables spinorielles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 118–129 (1942). [MF 13653]

Expository article in which the author follows closely work of Th. De Donder [Théorie Invariantive du Calcul des Variations, Gauthier-Villars, Paris, 1935] and finds identities which functions of spinors must satisfy if they are to be density factors in the strict sense. *A. Schwartz.*

Thermodynamics, Statistical Mechanics

Vlasov, A. On the theory of the solid state. Acad. Sci. USSR. J. Phys. 9, 130–138 (1945). [MF 13372]

This paper is based on the general theory of particle-ensembles, governed predominantly by collective-interaction-type forces as set forth in the author's earlier paper [Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] 8, 248–266 (1944); Acad. Sci. USSR. J. Phys. 9, 25–40 (1945); these Rev. 6, 222, 334; 7, 104]. The stress is here on the proof of the spontaneous emergence and the fundamental parameters of the solid crystalline state in the sense of Cauchy's original program (1828), as contrasted with Born's theory which postulates the existence of periodic structure a priori. As in the preceding paper, a linearized equation of state is obtained for the distribution function $f(x, y, z; \xi, \eta, \zeta)$ by considering the neighborhood of a stationary solution $\Phi_0(\xi, \eta, \zeta)$ of the original nonlinear equation, a solution which corresponds to the case of homogeneous density: $f=\Phi_0+\varphi(x, y, z; \xi, \eta, \zeta)$. The procedure differs somewhat from that adopted in the earlier paper in that the linearized part of the "close range" term $(\partial f/\partial t)^{st}$ is first retained as perturbation function:

$$\partial\varphi/\partial t+\mathbf{v}\,\mathrm{grad}_r\,\varphi-m^{-1}\,\mathrm{grad}_r\int K(|\mathbf{r}'-\mathbf{r}|)\times\int_{-\infty}^{\infty}\varphi d\omega d\tau'\,\mathrm{grad}_v\,\Phi_0=[\partial\Phi_0\varphi/\partial t]^{st},$$

where in a rough approximation $[\partial f/\partial t]^{st}$ may be understood as the Boltzmann term

$$\int_{-\infty}^{\infty}\int_0^r\int_0^{2\pi}(ff_1-f_1'f')vbdbd\omega d\epsilon,$$

where b and ϵ are collision parameters. By assuming Φ_0 to be symmetrical with respect to velocities and, more particularly, dependent on the particle energy ϵ only, the simpler equation

$$\partial\varphi/\partial t+\mathbf{v}\,\mathrm{grad}_r\left\{\varphi-\frac{\partial\Phi_0}{\partial\epsilon}\int_{-\infty}^{\infty}K(|\mathbf{r}'-\mathbf{r}|)\int_{-\infty}^{\infty}\varphi d\omega d\tau'\right\}=\left[\frac{\partial\Phi_0\varphi}{\partial t}\right]^{st}$$

is obtained.

This will allow a general solution by superposition of two solutions φ_1 and φ_2 of sharply different physical significance. The function φ_2 does not involve density changes and is shown to refer to particle ensembles without integral interaction, that is, gases. The function φ_1 satisfies the integral equation

$$\text{(A)}\qquad \varphi_1-(\partial\Phi_0/\partial\epsilon)\int_{-\infty}^{\infty}K(|\mathbf{r}'-\mathbf{r}|)\int_{-\infty}^{\infty}\varphi_1 d\omega d\tau'=0$$

and $[\partial\Phi_0\varphi_1/\partial t]^{st}=0$, the latter "collision" equation being automatically satisfied if Φ_0 is Maxwellian. The condition for spontaneous emergence of crystal structure is obtained from the discussion of the integral equation

$$\text{(B)}\qquad \rho(\mathbf{r})-\lambda\int_{-\infty}^{\infty}K_+(|\mathbf{r}'-\mathbf{r}|)\rho(\mathbf{r}')d\tau'=0,$$

which by

$$\lambda=-\int_{-\infty}^{\infty}(\partial\Phi_0/\partial\epsilon)d\xi d\eta d\zeta,\quad K_+(|\mathbf{r}'-\mathbf{r}|)=-K(|\mathbf{r}-\mathbf{r}'|),$$

$$\rho(\mathbf{r}')=\int_{-\infty}^{\infty}\varphi_1(\mathbf{r}', \mathbf{v}, t)d\omega$$

is equivalent to (A). This type of equation has, for one dimension, been studied by E. Hopf and L. Föppl in connection with problems of aerodynamics and elasticity. For the Maxwellian $\Phi_0=N(m/(2\pi kt))^{3/2}e^{-\epsilon/(kt)}$, we have $\lambda=N/(kt)$. It is found that periodic solutions, that is, crystalline states, do or do not exist according as $\lambda>\lambda_0$ or $\lambda<\lambda_0$, where $\lambda_0^{-1}=4\pi\int_0^\infty K_+(\rho)\rho^2d\rho$. This criterion does not necessarily call for positive forces of attraction $K_+(\rho)$ but stipulates merely that square-distance-weighted forces of attraction prevail. It is integral in nature, that is, not merely in terms of paired interactions. Whenever approximation in terms of short-range action is permissible, however, series expansion of $\rho(\mathbf{r}')$ in (B) leads to the simpler criterion $\lambda>1/k_0$ involving the zero moment $k_0=\int_{-\infty}^{\infty}K_+(|\mathbf{r}-\mathbf{r}'|)d\tau'$ and gives also the space period Λ_p of the structure:

$$(\Lambda_p/2\pi)^2=\lambda k_2/(\lambda k_0-1),$$

where

$$k_2=\tfrac{1}{2}\int_{-\infty}^{\infty}K_+(|\mathbf{r}-\mathbf{r}'|)(x-x')^2d\tau'.$$

The two quantities thus obtained are closely related to the radius of the sphere of action of the central force and to the "Debye distance" of the conventional theory. There follows a generalization concerning stability of crystal structure with several periods and a chapter on "Non-stationary equation. Kinetics of the process of crystallization." This deals with solutions of the progressive wave type and is based on the corresponding development in the paper quoted above. *H. G. Baerwald* (Cleveland Heights, Ohio).

Vlasov, A. A. On the theory of the solid state. Uchenye Zapiski Moskov. Gos. Univ. Fizika 77, 30–42 (1945). (Russian) [MF 15107]

Vlasov, A. A. On the problem of many bodies (vibrational properties, crystalline structure, nondissipative flows and spontaneous appearance of these properties in "gases"). Uchenye Zapiski Moskov. Gos. Univ. Fizika 77, 3–29 (1945). (Russian) [MF 15106]

Ribaud, G. Nouvelle expression du coefficient de convection de la chaleur en régime d'écoulement turbulent. J. Phys. Radium (8) 2, 12–25 (1941). [MF 13630]

The author modifies the treatments of Prandtl [Z. Verein. Deutsch. Ingenieure 77, 105–114 (1933)] and von Kármán [Trans. A.S.M.E. 61, 705–710 (1939)] to express the coefficient of heat transfer for turbulent flow by the formula

$$N_u/\Re P_r=\frac{f/2}{1+\frac{5}{3}(u'/u_0)(P_r^{\frac{2}{3}}-1)},$$

where u' is the stream velocity at the boundary of the laminar sublayer and u_0 is the maximum stream velocity. This formula is obtained by assuming the turbulent velocity

gradient near the boundary to be inversely proportional to the cube of the distance from the boundary. This assumption provides continuous velocity and temperature gradients and also permits a description of the velocity distribution in the laminar sublayer and the buffer layer by the single function,

$$ku^+ = \int_0^{ky^+} (1+x^4)^{-1}dx,$$

where u^+ and y^+ are the dimensionless turbulent velocity and distance parameters and k is the von Kármán constant, $k \cong .4$. *N. A. Hall* (New Hartford, Conn.).

Prigogine, I. Extension de l'équation de Saha au plasma non-isotherme. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 26, 53–63 (1940). [MF 13828]

Let T_γ denote the temperature of constituent γ, $\gamma=1, \cdots, c$ of a chemical system, the system being called non-isothermal if the T_γ's are not all equal. The author develops the relation

$$dE = \sum_\gamma T_\gamma dS_\gamma - PdV + \sum_\gamma \mu_\gamma' dn_\gamma$$

and uses this result to generalize the law of Guldberg and Waage. As a special case, he considers a strongly ionized gas in which the ions and neutral atoms have a common temperature T_1 while the free electrons have temperature T_2 and extends the equation of Saha to the form

$$\frac{x^{(T_1+T_2)/T_1}}{(1-x)(1+(T_2/T_1)x)^{T_2/T_1}} = \{P^{-1}h^{-3}(2\pi m_e)^{3/2}k^{5/2}T_1T_2^{3/2}\}^{T_2/T_1}(\pi_+/\pi_0)e^{-D/kT_1},$$

where x denotes the fractional part of the atoms that are ionized. A graphical method is given for determining x from this equation. *C. C. Torrance* (Cleveland, Ohio).

Braun, Alexandre. Comparaison des efforts tangentiels en fonction des flux de chaleur en divers régimes d'écoulement. C. R. Acad. Sci. Paris 220, 384–386 (1945). [MF 13952]

The author discusses the use of the form of the relation between unit shear at any point in a moving fluid and the heat transfer rate per unit area as a characterization of the nature of the local flow regime. *N. A. Hall.*

Kalikhman, L. E. Resistance and heat transfer of a plate in a flow of gas with high velocity. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 245–256 (1945). (Russian. English summary) [MF 14036]

The paper is concerned with the problem of the laminar and turbulent boundary layers of gas flow around a plane plate with heat exchange. The author derives the formulae for the resistance and heat transfer, distributions of velocities and temperatures, dissipation and heat content of the gas. The author also finds the limit numbers of Mach, beyond which the plate will lose its cooling action on the heated gas, or the gas will lose its cooling action upon the heated plate. *Author's summary.*

Gurevich, L. On the theory of thermal diffusion. Acad. Sci. USSR. J. Phys. 9, 312–316 (1945). [MF 14519]

In a mixture of several substances, let n_k, μ_k, α_k and β_k denote, respectively, the concentration, chemical potential, coefficient of thermal diffusion and coefficient of diffusional thermal conductivity for the substance of kind k. It is shown in this paper that $\beta_i = \sum_k T\alpha_k \partial\mu_k/\partial n_i$. For the case where the walls are of no importance, the author gives a general solution of the diffusion equations valid for non-stationary diffusion, and points out that previous calculations of nonstationary diffusion based on the ordinary diffusional equation "appear to be quantitatively incorrect, at least for gases." *C. C. Torrance* (Cleveland, Ohio).

Géhéniau, J. Les phénomènes de diffusion, dans la mécanique statistique de Th. De Donder. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 283–293 (1942). [MF 13663]

Starting from De Donder's generalization of the Hamiltonian equations, the author deduces equations governing the transfer of mass, momentum, energy, and entropy for the case of diffusing matter. *C. C. Torrance.*

Géhéniau, J. Les phénomènes de diffusion dans la mécanique statistique de Th. De Donder. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 29, 31–36 (1943). [MF 13837]

The author deduces the fundamental equations of diffusion for the case where there are no chemical phenomena [see the preceding review] and shows that the results of Meixner and Verschaffelt are compatible when they are properly interpreted [see the two following reviews]. *C. C. Torrance* (Cleveland, Ohio).

Verschaffelt, J. E. La thermomécanique de la diffusion des gaz. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 455–475 (1942). [MF 13666]

The author computes the flow of heat involved in a certain simple form of diffusion and claims to have found an error in a related formula of Meixner [Ann. Physik (5) 39, 333–356 (1941)]. *C. C. Torrance* (Cleveland, Ohio).

Verschaffelt, J. E. Sur la thermomécanique des fluides en mouvement. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 476–489 (1942). [MF 13667]

The author claims to have found another error in the results of Meixner [see the preceding review]. *C. C. Torrance* (Cleveland, Ohio).

Verschaffelt, J. E. Sur la thermomécanique de la conduction calorifique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 436–454 (1942). [MF 13665]

The author analyzes various methods for computing the time rate of change of entropy in matter where there is conduction and radiation of heat. *C. C. Torrance.*

***de Groot, S. R. L'Effet Soret. Diffusion Thermique dans les Phases Condensées.** N. V. Noord-Hollandsche Uitgevers Maatschappij, Amsterdam, 1945. 191 pp.

This volume gives a résumé of the experimental and theoretical work that has been carried out in the study of thermodiffusion in liquids and solids. Experimental techniques are described and outlines are given of phenomenological, thermodynamic and kinetic theories of thermodiffusion. *C. C. Torrance* (Cleveland, Ohio).

DeDonder, Th. La mécanique statistique relativiste. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 28, 240–246 (1942). [MF 13660]

This paper extends the author's previous results [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° 17, no. 5, 1–83 (1938)] to the simple relativistic Hamiltonian theory of motion of particles. The ideas and notation depend on the earlier paper. The methods are elementary: tensor notation and generalized Stokes-Green theorems. *B. O. Koopman* (New York, N. Y.).

non-
cula-
iffu-
t, at
).

éca-
Bel-
563]
amil-
ning
for
e.

éca-
Roy.
837]
iffu-
hena
s of
are

).

sion
455-

in a
ound
(5)
).

s en
28,

the

).

duc-
(5)

the
re is
e.

ique
sche

and
y of
ech-
eno-
mo-
).

cad.
42).

cad.
1-83
y of
the
tion

).